Heat Polynomial Analogues for Equations with Higher Order Time Derivatives

G. N. Hile
Mathematics Department
University of Hawaii
Honolulu, Hawaii 96822

Alexander Stanoyevitch
Mathematics Department
University of Guam
UOG Station, Mangilao, GU 96923

Corresponding author: Hile
hile@hawaii.edu, 808-956-6533, 808-956-9139 (fax)

Abstract

We generalize the heat polynomials for the heat equation to more general partial differential equations, of higher order with respect to both the time variable and the space variables. Whereas the heat equation requires only one family of polynomials, for an equation of the ℓ-th order with respect to time we introduce ℓ families of polynomials. These families correspond to the ℓ initial conditions specified by the Cauchy problem.

1 Introduction

The classical heat polynomials $\{p_\beta(x,t)\}$ are polynomial solutions of the heat equation,

$$\frac{\partial u(x,t)}{\partial t} = \Delta_n u(x,t) \quad ,$$

satisfying the initial condition

$$p_\beta(x,0) = x^\beta \quad . \tag{1}$$

These polynomials appear in early work of Appell [1] on the heat equation, and were later investigated in detail by Rosenbloom and Widder [28, 31, 32, 33]. The polynomials are particularly useful in solving the Cauchy problem

$$\frac{\partial u(x,t)}{\partial t} = \Delta u(x,t) \qquad , \qquad u(x,0) = f(x) \quad . \tag{2}$$

Rosenbloom and Widder showed that, if $f(x)$ obeys certain growth conditions and has the Taylor series expansion

$$f(x) = \sum_\beta c_\beta x^\beta \quad ,$$

then a solution of (2) is given by the series

$$u(x,t) = \sum_\beta c_\beta p_\beta(x,t) \quad .$$

In [33] Widder further demonstrated, with use of pointwise bounds on the heat polynomials, that a solution of the heat equation can be expanded in a series of these polynomials in the widest strip where it enjoys the Huygens property. Colton in [6] used these same bounds to demonstrate that the heat polynomials form a complete family of solutions of the heat equation in certain domains. In [17] Hile and Mawata used heat polynomial bounds to help describe the behaviour of the heat operator as a mapping between weighted Sobolev spaces in $\mathbb{R}^{n+1}$.

Here we present analogues of heat polynomials for an equation involving a higher order time derivative,

$$\frac{\partial^\ell u(x,t)}{\partial t^\ell} = \sum_\alpha a_\alpha \partial_x^\alpha u(x,t) \quad , \tag{3}$$

where ℓ may be any positive integer. Cauchy data for this equation involves ℓ initial conditions,

$$\frac{\partial^k u\,(x,0)}{\partial t^k} = f_k\,(x) \quad , \quad 0 \le k < \ell \quad , \tag{4}$$

and correspondingly we require k families of polynomials $\{p_{\beta,k}\,(x,t)\}$, $0 \le k < \ell$. The k-th family of polynomial solutions of (3) solves the Cauchy conditions

$$\frac{\partial^j}{\partial t^j} p_{\beta,k}\,(x,0) = \delta_{jk} x^\beta = \begin{cases} x^\beta & \text{, if } j = k \quad , \\ 0 & \text{, if } j \ne k \text{ and } 0 \le j < \ell \quad . \end{cases}$$

It turns out that, if each f_k in (4) has a Taylor expansion

$$f_k\,(x) = \sum_\beta c_{\beta,k} x^\beta \quad ,$$

then under certain growth conditions a solution u of the Cauchy problem (3) – (4) can be expressed as the superposition

$$u\,(x,t) = \sum_{k=0}^{\ell-1} \sum_\beta c_{\beta,k} p_{\beta,k}\,(x,t) \quad . \tag{5}$$

(As this latter development requires a rather lengthy and detailed derivation of pointwise bounds on the polynomials $\{p_\beta\}$, we postpone it along with other applications of these bounds for a later paper.)

There have been a number of generalizations of heat polynomials for equations besides the heat equation. Kemnitz [23] presented such polynomials for an equation in one space dimension,

$$\frac{\partial u}{\partial t} = \frac{\partial^r u}{\partial x^r} \qquad (r \ge 2) \quad . \tag{6}$$

Haimo and Markett [15, 16] studied polynomial solutions of a closely related equation, the so-called *higher order heat equation*,

$$\frac{\partial}{\partial t} u(x,t) = (-1)^{q+1} \frac{\partial^{2q}}{\partial x^{2q}} u(x,t) \quad . \tag{7}$$

When we specialize our equation (3) to the heat equation, or to (6) or to (7), our polynomials become the heat polynomials, or the polynomials of Kemnitz or of Haimo and Markett, respectively.

The *generalized heat equation*, or *radial heat equation*,

$$\frac{\partial u(r,t)}{\partial t} = \frac{\partial^2 u(r,t)}{\partial r^2} + \frac{2\nu}{r}\frac{\partial u}{\partial r} \quad , \tag{8}$$

has been investigated by Bragg [2] and more extensively by Haimo [8, 9, 10, 12, 13, 14]. (In the case $2\nu = n-1$ the right side of (8) is the n-dimensional Laplace operator in radial coordinates.) Cholewinski and Haimo [5] studied polynomial solutions of the equation

$$\frac{\partial u(x,t)}{\partial t} = x u_{xx}(x,t) + (\alpha + 1 - x)\, u_x(x,t) \quad ,$$

and in [11] Haimo did the same for an equation in n space variables,

$$\frac{\partial u(x,t)}{\partial t} = \Delta_n u(x,t) + \sum_{i=1}^{n} \frac{2\nu}{x_i}\frac{\partial u(x,t)}{\partial x_i} \quad .$$

Fitouhi [7] extended the theory of heat polynomials to a slightly more general version of (8),

$$u_t(x,t) = u_{xx}(x,t) + \left(\frac{2\nu}{x} + \frac{B'(x)}{B(x)}\right) u_x(x,t) \quad ,$$

with B a suitable analytic function. As the coefficients of all these "generalized heat equations" depend on the space variable x, our theory does not directly apply to them.

Lo [24] presented analogues of the heat polynomials, called *generalized Helmholtz polynomials*, for a perturbed heat equation in one space variable,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \varepsilon^2 \frac{\partial^2 u}{\partial t^2} \quad . \tag{9}$$

Lo's polynomials have properties much like those of the ordinary heat polynomials, and when $\varepsilon = 0$ they reduce to the heat polynomials. As (9) does not quite fit into our format, likewise our theory does not apply to this equation.

The present authors in [18, 19] established explicit formulas for polynomial solutions $\{p_\beta\}$, satisfying the initial condition (1), of a general evolution equation

$$\frac{\partial u\,(x,t)}{\partial t} = \sum_{\alpha} a_\alpha\,(t)\,\partial_x^\alpha u\,(x,t) \quad . \tag{10}$$

They also investigated properties of these polynomials, derived pointwise upper bound estimates, and used these to solve Cauchy problems via series expansions in terms of the polynomials.

There is an interesting body of work on the problem of determining all polynomial solutions of systems of partial differential equations with constant coefficients, as well as the dimensions of solution spaces of polynomials of specified degree. For important papers and further bibliographical references, see [20, 21, 22, 25, 26, 27, 30].

Another interesting application of the heat polynomials appears in the work of L. R. Bragg and J. W. Dettman concerning transmutation operators. (See [4] for a list of references.) For example, in [3] these authors demonstrate that various polynomial solutions of elliptic and hyperbolic equations can be obtained from the heat polynomials, and that these in turn can be used to represent solutions of problems involving these equations.

The authors thank the referee for pointing out representation (34) for our polynomials.

2 The Equation

Let $\mathcal{L}$ denote the linear differential operator, with constant coefficients $\{a_\alpha\}$,

$$\mathcal{L}u = \sum_{\alpha} a_\alpha \partial_x^\alpha u \quad . \tag{11}$$

Given a positive integer ℓ, we consider the differential equation

$$\frac{\partial^\ell u}{\partial t^\ell} = \mathcal{L}u \quad . \tag{12}$$

We consider solutions $u = u(x,t) = u(x_1, x_2, \cdots, x_n, t)$ of this equation defined for $(x,t) \in \mathbb{R}^n \times \mathbb{R}$. Although in most applications u is real valued and the coefficients $\{a_\alpha\}$ are real, our analysis is equally valid for complex valued u and complex $\{a_\alpha\}$; thus we allow this more general setting. We shall refer to t as the "time variable" and to x as the "space variable", although this distinction is somewhat arbitrary as these physical interpretations do not pertain to all equations. For multi-indices $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ in $\mathbb{R}^n$ we adopt the usual notation

$$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n \quad , \quad \alpha! = \alpha_1!\alpha_2! \cdots \alpha_n! \quad , \quad x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \quad .$$

We let ∂_x^α denote the "space derivative"

$$\partial_x^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}} \quad .$$

We assume the summation in (11) is over only a finite number of multi-indices α. At times it is convenient to number these multi-indices as α^1, α^2, $\cdots$, α^I, and the corresponding coefficients $\{a_\alpha\}$ as a_1, a_2, $\cdots$, a_I, so that $\mathcal{L}$ may be written in the alternate formulation

$$\mathcal{L}u = \sum_{i=1}^{I} a_i \partial_x^{\alpha^i} u \quad . \tag{13}$$

3 The Function E

We will define generating functions for polynomial solutions of (12) in terms of a more elementary function,

$$E\left(t, s; \ell, k\right) = \sum_{m=0}^{\infty} \frac{t^{\ell m + k} s^m}{(\ell m + k)!} \quad . \tag{14}$$

We view ℓ and k as parameters of this function, and t and s as the independent variables. We restrict ℓ to belong to the set $\mathbb{N}$ of positive integers, and k to the set $\mathbb{N}_0$ of nonnegative integers. We allow s to be any complex number, although in most applications it will be real. In order to avoid complex derivatives we restrict t to be real, although much of what we do is equally valid if t is complex. It is clear that the power series (14) converges for all values of $(t, s, \ell, k) \in \mathbb{R} \times \mathbb{C} \times \mathbb{N} \times \mathbb{N}_0$, representing a C^∞ function with respect to the variables t and s, and that termwise differentiation of all orders with respect to t and/or s valid. The function E might be regarded as somewhat of a generalized exponential function. Observe that

$$\begin{aligned} E\left(t, 1; 1, 0\right) &= e^t \quad , & E\left(t, s; 1, 0\right) &= e^{st} \quad , \\ E\left(t, 1; 2, 0\right) &= \cosh t \quad , & E\left(t, 1; 2, 1\right) &= \sinh t \quad , \\ E\left(t, -1; 2, 0\right) &= \cos t \quad , & E\left(t, -1; 2, 1\right) &= \sin t \quad . \end{aligned}$$

From (14) it follows that, for $k = 1, 2, 3, \cdots$,

$$\frac{\partial}{\partial t} E\left(t, s; \ell, k\right) = E\left(t, s; \ell, k-1\right) \quad , \quad \int_0^t E\left(r, s; \ell, k-1\right) \, dr = E\left(t, s; \ell, k\right) \; .$$

We are mainly interested in integral values of ℓ and k such that $0 \leq k < \ell$. Writing (14) in the expanded form

$$E\left(t, s; \ell, k\right) = \left[\frac{t^k}{k!} + \frac{t^{\ell + k} s}{(\ell + k)!} + \frac{t^{2\ell + k} s^2}{(2\ell + k)!} + \frac{t^{3\ell + k} s^3}{(3\ell + k)!} + \cdots \right] \quad ,$$

we determine readily that, for $0 \le k < \ell$,

$$\frac{\partial^j}{\partial t^j} E\left(0, s; \ell, k\right) = \begin{cases} 1 & \text{, if } j = k \quad , \\ 0 & \text{, if } j \neq k \text{ and } 0 \le j < \ell \quad . \end{cases} \tag{15}$$

Moreover, also for $0 \le k < \ell$,

$$\frac{\partial^\ell}{\partial t^\ell} E\left(t, s; \ell, k\right) = \sum_{m=1}^{\infty} \frac{t^{\ell m + k - \ell} s^m}{(\ell m + k - \ell)!} = \sum_{m=0}^{\infty} \frac{t^{\ell m + k} s^{m+1}}{(\ell m + k)!} \quad ,$$

and thus

$$\frac{\partial^\ell}{\partial t^\ell} E\left(t, s; \ell, k\right) = sE\left(t, s; \ell, k\right) \qquad , \quad 0 \le k < \ell \quad . \tag{16}$$

4 The Polynomials

It is useful to designate a "vector of coefficients", associated with $\mathcal{L}$ and more specifically with its alternate form (13),

$$a = (a_1, a_2, \cdots, a_I) \quad , \tag{17}$$

as well as an associated polynomial $Q = Q\left(y\right)$, $y \in \mathbb{R}^n$,

$$Q\left(y\right) = \sum_{\alpha} a_\alpha y^\alpha = \sum_{i=1}^{I} a_i y^{\alpha^i} \quad . \tag{18}$$

Given a positive integer ℓ, we employ ℓ different "generating functions" associated with the operator $\mathcal{L}$. These are labelled as $\{G_k\}$, $k = 0, 1, \cdots, \ell - 1$, and defined for $(x, t, y) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$ according to

$$\begin{aligned} G_k\left(x, t, y\right) &= e^{x \cdot y} E\left(t, Q\left(y\right); \ell, k\right) & (19) \\ &= e^{x \cdot y} \sum_{m=0}^{\infty} \frac{t^{\ell m + k} Q\left(y\right)^m}{(\ell m + k)!} \qquad (0 \le k < \ell) \quad . \end{aligned}$$

(It would perhaps be more precise to subscript G as $G_{\ell,k}$, but to make the notation less cluttered we regard ℓ as given and fixed.) The notation $x \cdot y$ denotes the usual dot product in $\mathbb{R}^n$. We recall that

$$e^{x \cdot y} = \sum_{\gamma} \frac{x^\gamma y^\gamma}{\gamma!} \quad , \tag{20}$$

with the summation over all multi-indices $\gamma = (\gamma_1, \gamma_2, \cdots, \gamma_n)$ in $\mathbb{R}^n$.

We examine some properties of the generating functions $\{G_k\}$. An immediate consequence of (19) is the pair of formulas, valid for $1 \leq k < \ell$,

$$\frac{\partial}{\partial t} G_k(x,t,y) = G_{k-1}(x,t,y) \qquad , \qquad \int_0^t G_{k-1}(x,r,y)\ dr = G_k(x,t,y) \quad .$$

From (15) we infer, also for $0 \leq k < \ell$,

$$\frac{\partial^j G_k(x,0,y)}{\partial t^j} = \begin{cases} e^{x\cdot y} & , \text{if } j = k \quad , \\ 0 & , \text{if } j \neq k \text{ and } 0 \leq j < \ell \ . \end{cases} \tag{21}$$

Moreover, from (16) and (19),

$$\frac{\partial^\ell}{\partial t^\ell} G_k(x,t,y) = Q(y)\, G_k(x,t,y) = \sum_\alpha a_\alpha \partial_x^\alpha G_k(x,t,y) \quad ;$$

that is, for each fixed y in $\mathbb{R}^n$,

$$\frac{\partial^\ell}{\partial t^\ell} G_k(x,t,y) = \mathcal{L} G_k(x,t,y) \qquad , \qquad 0 \leq k < \ell \ . \tag{22}$$

We shall write each G_k in the expanded form

$$G_k(x,t,y) = \sum_\beta p_{\beta,k}(x,t)\, \frac{y^\beta}{\beta!} \quad , \tag{23}$$

where for each $k = 0, 1, \cdots, \ell - 1$, the collection $\{p_{\beta,k}(x,t)\}$ is a family of functions indexed by multi-indices β in $\mathbb{R}^n$.

We recall the general multinomial formula,

$$(c_1 + c_2 + \cdots + c_I)^m = \sum_{|\sigma| = m} \frac{m!}{\sigma!}\, c^\sigma \quad , \tag{24}$$

where $c = (c_1, c_2, \cdots, c_I)$, $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_I)$, and the summation is over all multi-indices σ in R^I of magnitude m. With this formula, (18) leads to

$$\begin{aligned} Q(y)^m = \left(\sum_{i=1}^I a_i y^{\alpha^i} \right)^m &= \sum_{|\sigma|=m} \frac{m!}{\sigma!} \left(a_1 y^{\alpha^1}, a_2 y^{\alpha^2}, \cdots, a_I y^{\alpha^I} \right)^\sigma \\ &= \sum_{|\sigma|=m} \frac{m!}{\sigma!}\, a_1^{\sigma_1} a_2^{\sigma_2} \cdots a_n^{\sigma_n}\ y^{\sigma_1 \alpha^1 + \sigma_2 \alpha^2 + \cdots + \sigma_I \alpha^I} \quad . \end{aligned}$$

We define a "vector of multi-indices"

$$\overline{\alpha} = \left(\alpha^1, \alpha^2, \cdots, \alpha^I\right) \quad ,$$

and introduce a "dot product"

$$\overline{\alpha} \cdot \sigma = \sigma_1 \alpha^1 + \sigma_2 \alpha^2 + \cdots + \sigma_I \alpha^I \quad . \tag{25}$$

Then $Q(y)^m$ can be written more briefly as

$$Q(y)^m = \sum_{|\sigma|=m} \frac{m!}{\sigma!}\, a^\sigma \, y^{\overline{\alpha}\cdot\sigma} \quad .$$

(Note that $\overline{\alpha}\cdot\sigma$ is a multi-index in $\mathbb{R}^n$.) Substitution of this expression into (19), with use also of (20), gives

$$\begin{aligned} G_k(x,t,y) &= \sum_\gamma \frac{x^\gamma y^\gamma}{\gamma!} \sum_{m=0}^\infty \frac{t^{\ell m + k}}{(\ell m + k)!} \sum_{|\sigma|=m} \frac{m!}{\sigma!}\, a^\sigma \, y^{\overline{\alpha}\cdot\sigma} \\ &= \sum_\gamma \sum_\sigma \frac{x^\gamma y^\gamma}{\gamma!} \frac{t^{\ell|\sigma|+k}}{(\ell|\sigma|+k)!} \frac{|\sigma|! a^\sigma y^{\overline{\alpha}\cdot\sigma}}{\sigma!} \quad , \end{aligned} \tag{26}$$

with the summation over all multi-indices γ in $\mathbb{R}^n$ and σ in $\mathbb{R}^I$.

We want to fully justify any rearrangements and termwise differentiations of the double series (26) for G_k. We look at a compact region where $|x|, |y|, |t| \leq M$, with $M \geq 1$. Applying estimates of the type $|x^\alpha| \leq |x|^{|\alpha|}$, for each term of (26) we have the bound

$$\left| \frac{x^\gamma y^\gamma}{\gamma!} \frac{t^{\ell|\sigma|+k}}{(\ell|\sigma|+k)!} \frac{|\sigma|! a^\sigma y^{\overline{\alpha}\cdot\sigma}}{\sigma!} \right| \leq \frac{M^{|\gamma|} M^{|\gamma|}}{\gamma!} \frac{|\sigma|!}{(\ell|\sigma|+k)!} \frac{M^{\ell|\sigma|+k}\, |a|^{|\sigma|}\, M^{|\overline{\alpha}\cdot\sigma|}}{\sigma!} \quad .$$

Let L denote the maximum order of any space derivative in (11), so that $|\alpha| \leq L$ for each α. Then

$$|\overline{\alpha}\cdot\sigma| = \sum_{i=1}^I \left|\alpha^i\right| \sigma_i \leq L \sum_{i=1}^I \sigma_i = L|\sigma| \quad , \quad M^{|\overline{\alpha}\cdot\sigma|} \leq M^{L|\sigma|} \quad .$$

We use also the crude estimate

$$\frac{|\sigma|!}{(\ell|\sigma|+k)!} \leq \frac{1}{k!} \quad ,$$

along with the general formula

$$\sum_{\beta} \frac{r^{|\beta|}}{\beta!} = e^{nr} \quad ,$$

valid for real numbers r with the sum over multi-indices β in $\mathbb{R}^n$. We find that (26) is majorized by

$$\begin{aligned}
&\sum_{\gamma} \sum_{\sigma} \left| \frac{x^{\gamma} y^{\gamma}}{\gamma!} \frac{t^{\ell|\sigma|+k}}{(\ell\,|\sigma| + k)!} \frac{|\sigma|! a^{\sigma} y^{\overline{\alpha}\cdot\sigma}}{\sigma!} \right| \\
&\leq \frac{M^k}{k!} \sum_{\gamma} \frac{M^{2|\gamma|}}{\gamma!} \sum_{\sigma} \frac{|a|^{|\sigma|} M^{\ell|\sigma|} M^{L|\sigma|}}{\sigma!} \\
&\leq \exp M \exp\left(nM^2\right) \exp\left(I\,|a|\,M^{\ell+L}\right) \quad .
\end{aligned}$$

Thus (26) converges absolutely and uniformly in any compact region containing the variables x, t, and y.

With similar estimates we can verify that any differentiated series of (26), of any order and with respect to any combination of components of the variables x, t, y, likewise converges absolutely and uniformly in any compact region. (Any differentiated series will be majorized by a series comparable to an exponential series via the ratio test.) In particular, orders of summation can be freely interchanged in (26), and termwise differentiation is legitimate.

Now we interchange orders of summation in (26), first summing outside over powers y^{β}, as $\beta = \gamma + \overline{\alpha} \cdot \sigma$ ranges over all multi-indices in $\mathbb{R}^n$, getting

$$G_k\left(x, t, y\right) = \sum_{\beta} y^{\beta} \sum_{\gamma+\overline{\alpha}\cdot\sigma=\beta} \frac{|\sigma|!}{\sigma!} \frac{a^{\sigma} x^{\gamma} t^{\ell|\sigma|+k}}{\gamma!\,(\ell\,|\sigma| + k)!} \quad . \tag{27}$$

Comparing (27) with (23), we find that our formula for $p_{\beta,k}$ is

$$p_{\beta,k}\left(x, t\right) = \beta! \sum_{\gamma+\overline{\alpha}\cdot\sigma=\beta} \frac{|\sigma|!}{\sigma!} \frac{a^{\sigma} x^{\gamma} t^{\ell|\sigma|+k}}{\gamma!\,(\ell\,|\sigma| + k)!} \qquad (0 \leq k < \ell) \quad . \tag{28}$$

Given multi-indices α and β in $\mathbb{R}^n$ we say that $\alpha \leq \beta$ provided that $\alpha_i \leq \beta_i$ for each i. The sum in (28) is over multi-indices γ in $\mathbb{R}^n$ and σ in $\mathbb{R}^I$ satisfying the condition $\gamma + \alpha \cdot \sigma = \beta$. But the only way this condition is possible is that $\overline{\alpha} \cdot \sigma \leq \beta$ and $\gamma = \beta - \overline{\alpha} \cdot \sigma$, and for any such σ there is only one corresponding γ. Thus we may rewrite (28) in the equivalent formulation

$$p_{\beta,k}(x,t) = \beta! \sum_{\overline{\alpha}\cdot\sigma\leq\beta} \frac{|\sigma|!}{\sigma!} \frac{a^{\sigma} x^{\beta-\overline{\alpha}\cdot\sigma} t^{\ell|\sigma|+k}}{(\beta-\overline{\alpha}\cdot\sigma)!\,(\ell\,|\sigma|+k)!} \qquad (0\leq k<\ell) \quad , \tag{29}$$

with the summation now taken over all multi-indices σ in $\mathbb{R}^I$ such that $\overline{\alpha}\cdot\sigma\leq\beta$.

As termwise differentiation is permissible, (22) and (23) give

$$\sum_{\beta} \frac{\partial^{\ell}}{\partial t^{\ell}} p_{\beta,k}(x,t)\,\frac{y^{\beta}}{\beta!} = \frac{\partial^{\ell}}{\partial t^{\ell}} G_k(x,t,y) = \mathcal{L} G_k(x,t,y) = \sum_{\beta} \mathcal{L} p_{\beta,k}(x,t)\,\frac{y^{\beta}}{\beta!} \ .$$

Hence

$$\frac{\partial^{\ell}}{\partial t^{\ell}} p_{\beta,k}(x,t) = \mathcal{L} p_{\beta,k}(x,t) \quad . \tag{30}$$

Also, from (23) and (21), if $0\leq k<\ell$ then

$$\begin{aligned} \sum_{\beta} \left[\frac{\partial^{j}}{\partial t^{j}} p_{\beta,k}(x,0)\right] \frac{y^{\beta}}{\beta!} &= \frac{\partial^{j} G_k(x,0,y)}{\partial t^{j}} \\ &= \begin{cases} e^{x\cdot y} = \sum_{\beta} \frac{x^{\beta} y^{\beta}}{\beta!} & , \text{ if } j=k \quad , \\ 0 & , \text{ if } j\neq k \text{ and } 0\leq j<\ell \quad . \end{cases} \end{aligned}$$

We conclude that, for $0\leq k<\ell$,

$$\frac{\partial^{j}}{\partial t^{j}} p_{\beta,k}(x,0) = \begin{cases} x^{\beta} & , \text{ if } j=k \quad , \\ 0 & , \text{ if } j\neq k \text{ and } 0\leq j<\ell \quad . \end{cases} \tag{31}$$

As there are only a finite number of powers x^{γ} with $\gamma\leq\beta$, (28) shows that $p_{\beta,k}(x,t)$ is a polynomial in x when t is held fixed. If the operator $\mathcal{L}$ of (11) contains a *zero order term* – that is, if some nonzero a_{α} appears corresponding to $\alpha=(0,\cdots,0)$ – then there will be an infinite number of multi-indices σ in $\mathbb{R}^I$ with $\overline{\alpha}\cdot\sigma\leq\beta$. In this case the summations in (28) and (29) will have an infinite number of terms. (A term $\sigma_i\alpha^i$ in (25) contributes nothing to the sum if $\alpha^i=(0,\cdots,0)$.) Thus, if $\mathcal{L}$ has a zero order term, $p_{\beta,k}(x,t)$ need not be a polynomial with respect to t. However, if $\mathcal{L}$ has no zero order term then the number of multi-indices σ with $\overline{\alpha}\cdot\sigma\leq\beta$ is finite, and (28) and (29) both are finite summations. In this event $p_{\beta,k}(x,t)$ is a polynomial in both x and t – that is, it is a polynomial in the $n+1$ variables $(x_1,\cdots,x_n,t)$.

We summarize formally the results thus far of this section :

Theorem 1 *Let $\mathcal{L}$ be the operator (11) with real or complex constant coefficients $\{a_\alpha\}$, and let ℓ be a positive integer. Then the functions $\{p_{\beta,k}(x,t)\}$, as defined by (28) or the equivalent (29), and indexed by multi-indices β in $\mathbb{R}^n$ and integers k, $0 \le k < \ell$, solve the initial value problems*

$$\frac{\partial^\ell}{\partial t^\ell} p_{\beta,k}(x,t) = \mathcal{L} p_{\beta,k}(x,t) \quad ,$$

$$\frac{\partial^j}{\partial t^j} p_{\beta,k}(x,0) = \begin{cases} x^\beta & , \text{ if } j = k \quad , \\ 0 & , \text{ if } j \neq k \text{ and } 0 \le j < \ell \quad . \end{cases}$$

Each function $p_{\beta,k}$ is a polynomial in x when t is held fixed, and is a polynomial in both x and t in the event that $\mathcal{L}$ has no zero order term.

From (28) it follows that, for $1 \le k < \ell$,

$$\frac{\partial}{\partial t} p_{\beta,k}(x,t) = p_{\beta,k-1}(x,t) \quad , \quad \int_0^t p_{\beta,k-1}(x,r)\ dr = p_{\beta,k}(x,t) \quad . \tag{32}$$

Given any multi-index γ in $\mathbb{R}^n$, we may differentiate (23) and (19) to obtain

$$\begin{aligned} \sum_\beta \partial_x^\gamma p_{\beta,k}(x,t) \frac{y^\beta}{\beta!} &= \partial_x^\gamma G_k(x,t,y) = y^\gamma G_k(x,t,y) \\ &= y^\gamma \sum_\beta p_{\beta,k}(x,t) \frac{y^\beta}{\beta!} = \sum_\beta p_{\beta,k}(x,t) \frac{y^{\beta+\gamma}}{\beta!} \\ &= \sum_{\beta \ge \gamma} p_{\beta-\gamma,k}(x,t) \frac{y^\beta}{(\beta-\gamma)!} \quad . \end{aligned}$$

Consequently,

$$\partial_x^\gamma p_{\beta,k}(x,t) = \begin{cases} \frac{\beta!}{(\beta-\gamma)!}\, p_{\beta-\gamma,k}(x,t) & , \text{ if } \beta \ge \gamma \quad . \\ 0 & , \text{ if } \beta \not\ge \gamma \quad . \end{cases} \tag{33}$$

There is still another interesting way to present the polynomials $\{p_\beta\}$. With use of the identity

$$\partial^\gamma \left(x^\beta\right) = \begin{cases} \beta!/(\beta-\gamma)!\, x^{\beta-\gamma} & , \text{ if } \gamma \le \beta, \\ 0 & , \text{ otherwise,} \end{cases}$$

we may write (29) as

$$p_{\beta,k}(x,t) = \sum_{\overline{\alpha}\cdot\sigma \le \beta} \frac{|\sigma|!}{\sigma!} \frac{a^\sigma t^{\ell|\sigma|+k}}{(\ell|\sigma|+k)!}\, \partial^{\overline{\alpha}\cdot\sigma}\left(x^\beta\right) \quad .$$

But since $\partial^{\overline{\alpha}\cdot\sigma}\left(x^{\beta}\right)=0$ when $\overline{\alpha}\cdot\sigma$ is false, we may in fact sum over *all* σ in $\mathbb{R}^I$ to obtain

$$p_{\beta,k}\left(x,t\right)=\sum_{m=0}^{\infty}\frac{t^{\ell m+k}}{(\ell m+k)!}\sum_{|\sigma|=m}\frac{m!}{\sigma!}a^{\sigma}\partial^{\overline{\alpha}\cdot\sigma}\left(x^{\beta}\right)\quad .$$

Now, using (13) and (24), we observe that

$$\mathcal{L}^m=\left(\sum_{i=1}^{I}a_i\partial_x^{\alpha^i}\right)^m=\sum_{|\sigma|=m}\frac{m!}{\sigma!}a^{\sigma}\partial^{\overline{\alpha}\cdot\sigma}\quad .$$

Thus our formula for $p_{\beta,k}$ becomes

$$p_{\beta,k}\left(x,t\right)=\sum_{m=0}^{\infty}\frac{t^{\ell m+k}}{(\ell m+k)!}\mathcal{L}^m\left(x^{\beta}\right)\quad . \tag{34}$$

Use of equation (14) allows the brief symbolic representation

$$p_{\beta,k}\left(x,t\right)=E\left(t,\mathcal{L};\ell,k\right)x^{\beta}\qquad \left(0\leq k<\ell\right)\quad . \tag{35}$$

Note that if $\mathcal{L}$ has no zero order term then (34) terminates after a finite number of terms, yielding a polynomial solution.

If in (34) we designate functions

$$u_m\left(x,t\right)=\frac{t^{\ell m+k}}{(\ell m+k)!}\,x^{\beta}\quad ,\quad m=0,1,2,3,\ldots,$$

then we can write this formula as

$$p_{\beta,k}\left(x,t\right)=\sum_{m=0}^{\infty}\mathcal{L}^m\left[u_m\left(x,t\right)\right]\quad . \tag{36}$$

Moreover, the sequence of functions $\{u_m\}$ has the properties

$$\frac{d^{\ell}}{dt^{\ell}}\,u_m\left(x,t\right)=\begin{cases}0 & \text{, if } m=0,\\ u_{m-1}\left(x,t\right) & \text{, if } m\geq 1.\end{cases}$$

In the terminology of Karachik [20, 22], the sequence $\{u_m\}$ is "0-normalized with respect to the operator d^{ℓ}/dt^{ℓ}". Karachik shows that, for a large class of constant coefficient partial differential equations of the form

$$\mathcal{K}u=\mathcal{M}u-\mathcal{L}u=0\quad ,$$

polynomial solutions can be written in the form (36) where $\{u_m\}$ is 0-normalized with repect to the operator $\mathcal{M}$. Representation (34) is but one example of this general formulation.

5 Examples

We give a few examples of equations having the form (12), and of the associated polynomials.

Example 2 *If we take $\ell = 1$ and $\mathcal{L} = \Delta$, then (12) becomes the heat equation in n space variables,*

$$\frac{\partial u}{\partial t} = \Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} \quad .$$

We have $a = (1, \cdots, 1)$ and $\overline{\alpha} = (2e_1, \cdots, 2e_n)$, where e_i denotes the i-th unit coordinate vector. Also, $\sigma = (\sigma_1, \cdots, \sigma_n)$ and $\overline{\alpha} \cdot \sigma = 2\sigma_1 e_1 + \cdots + 2\sigma_n e_n = 2\sigma$. In formula (29) we must take $k = 0$, and thus we obtain but one family of polynomials,

$$p_\beta(x,t) = \beta! \sum_{2\sigma \le \beta} \frac{x^{\beta - 2\sigma} t^{|\sigma|}}{(\beta - 2\sigma)! \sigma!} \quad .$$

These are the classical n-dimensional heat polynomials studied by Rosenbloom and Widder [28, 31, 32].

Example 3 *With $\ell = 1$ and in one space dimension $n = 1$, and with $\mathcal{L} = \partial^r / \partial x^r$, $r \ge 1$, (12) becomes the equation studied by Kemnitz [23],*

$$\frac{\partial u}{\partial t} = \frac{\partial^r u}{\partial x^r} \quad . \tag{37}$$

Formula (28) specializes to the formula of Kemnitz,

$$p_\beta(x,t) = \beta! \sum_{\gamma + r\sigma = \beta} \frac{x^\gamma t^\sigma}{\gamma! \sigma!} \quad ,$$

where in one space dimension the multi-indices β, γ, and σ reduce to nonnegative integers. Haimo and Markett [15, 16] studied polynomial solutions of a closely related equation,

$$\frac{\partial u}{\partial t} = (-1)^{m+1} \frac{\partial^{2m} u}{\partial x^{2m}} \quad .$$

For this equation, (29) specializes to the formula of Haimo and Markett,

$$p_\beta(x,t) = \beta! \sum_{2m\sigma \le \beta} (-1)^{(m+1)\sigma} \frac{x^{\beta - 2m\sigma} t^\sigma}{\sigma! (\beta - 2m\sigma)!} \quad ,$$

where once again β and σ are nonnegative integers.

Example 4 *A natural extension of the Kemnitz equation (37) is the equation in one space dimension,*

$$\frac{\partial^\ell u}{\partial t^\ell} = \frac{\partial^r u}{\partial x^r} \quad , \tag{38}$$

where ℓ and r are positive integers. (The case $\ell = r = 4$, for example, is investigated in [29].) Here the multi-indices β, γ, and σ, all one-dimensional, may be viewed as nonnegative integers. The only space multi-index α involved in the equation is $\alpha^1 = (r)$, with the corresponding coefficient $a_1 = 1$. Formulas (28 – 29) produce the polynomial representations

$$p_{\beta,k}(x,t) = \beta! \sum_{\gamma + r\sigma = \beta} \frac{x^\gamma t^{\ell\sigma + k}}{\gamma!\,(\ell\sigma + k)!} = \beta! \sum_{r\sigma \le \beta} \frac{x^{\beta - r\sigma} t^{\ell\sigma + k}}{(\beta - r\sigma)!\,(\ell\sigma + k)!} \quad . \tag{39}$$

The first summation is taken over all γ and σ such that $\gamma + r\sigma = \beta$, while the second is taken over all σ such that $r\sigma \le \beta$. For example, if $\ell = r = 4$ then the polynomial $p_{10,1}(x,t)$ is

$$p_{10,1}(x,t) = 10! \sum_{4\sigma \le 10} \frac{x^{10-4\sigma} t^{4\sigma+1}}{(10-4\sigma)!\,(4\sigma+1)!} = x^{10}t + 42x^6t^5 + 5x^2t^9 \quad .$$

This polynomial solves (38) with $\ell = r = 4$, and with Cauchy data

$$p(x,0) = 0 \quad , \quad p_t(x,0) = x^{10} \quad , \quad p_{tt}(x,0) = 0 \quad , \quad p_{ttt}(x,0) = 0 \quad .$$

Note that we can write (39) also as

$$p_{\beta,k}(x,t) = \sum_{\sigma=0}^{\infty} \frac{t^{\ell\sigma+k}}{(\ell\sigma+k)!} \left(\frac{\partial^r}{\partial x^r}\right)^\sigma \left(x^\beta\right) \quad ,$$

thereby producing a simple illustration of the alternative formula (34).

Example 5 *If we take $\ell = 1$ but leave L as in (11), then (12) simplifies to an equation studied by the present authors in [18, 19]. Again, in (29) we must take $k = 0$, obtaining the single family of polynomials*

$$p_\beta(x,t) = \beta! \sum_{\overline{\alpha}\cdot\sigma \le \beta} \frac{a^\sigma x^{\beta - \overline{\alpha}\cdot\sigma} t^{|\sigma|}}{\sigma!\,(\beta - \overline{\alpha}\cdot\sigma)!} \quad .$$

Each polynomial p_β solves the Cauchy problem

$$\frac{\partial}{\partial t} p_\beta(x,t) = \mathcal{L} p_\beta(x,t) \qquad , \qquad p_\beta(x,0) = x^\beta \quad .$$

Example 6 *The complex Cauchy-Riemann equation for analytic functions is*

$$f_x + i f_y = 0 \quad ,$$

satisfied by analytic functions $f = f(x,y)$. We write this equation in the form

$$f_y = i f_x \quad ,$$

and substitute into (28) with t replaced by y. We have $\ell = 1$, $k = 0$, and all multi-indices are scalars, with $\overline{\alpha} \cdot \sigma = \sigma$, $a^{\sigma} = i^{\sigma}$. Writing $p_{\beta}(x,y) = p_{\beta,0}(x,y)$, we find that

$$p_{\beta}(x,y) = \beta! \sum_{\gamma+\sigma=\beta} \frac{i^{\sigma} x^{\gamma} y^{\sigma}}{\gamma! \sigma!} = \sum_{\gamma=0}^{\beta} \binom{\beta}{\gamma} x^{\gamma} (iy)^{\beta-\gamma} = (x+iy)^{\beta} \quad .$$

These analytic polynomials, satisfing the initial conditions $p_{\beta}(x,0) = x^{\beta}$, are of course the familiar powers z^{β} of the complex variable $z = x + iy$.

Example 7 *For the wave equation in n space dimensions,*

$$\frac{\partial^2 u}{\partial t^2} = \Delta_n u \quad ,$$

we have $\ell = 2$. Then (28) gives two families of polynomials,

$$p_{\beta,0}(x,t) = \beta! \sum_{\gamma+2\sigma=\beta} \frac{|\sigma|!}{(2|\sigma|)!} \frac{x^{\gamma} t^{2|\sigma|}}{\gamma! \sigma!} \quad ,$$
$$p_{\beta,1}(x,t) = \beta! \sum_{\gamma+2\sigma=\beta} \frac{|\sigma|!}{(2|\sigma|+1)!} \frac{x^{\gamma} t^{2|\sigma|+1}}{\gamma! \sigma!} \quad ,$$

where the sum is over multi-indices γ, σ in $\mathbb{R}^n$. These polynomials solve the wave equation, with initial conditions

$$p_{\beta,0}(x,0) = x^{\beta} \quad , \quad p_{\beta,1}(x,0) = 0 \quad ,$$
$$\frac{\partial}{\partial t} p_{\beta,0}(x,0) = 0 \quad , \quad \frac{\partial}{\partial t} p_{\beta,1}(x,0) = x^{\beta} \quad .$$

In the case of one space dimension $n = 1$, both γ and σ are nonnegative integers, and it can be verified that the polynomials simplify to

$$p_{\beta,0}(x,t) = \frac{1}{2} \left[(x+t)^{\beta} + (x-t)^{\beta} \right] \quad ,$$
$$p_{\beta,1}(x,t) = \frac{1}{2(\beta+1)} \left[(x+t)^{\beta+1} - (x-t)^{\beta+1} \right] \quad .$$

Example 8 *The equation governing the two-dimensional free transverse vibration of a thin elastic plate is*

$$\frac{\partial^2 u\left(x,y,t\right)}{\partial t^2}+K^2\Delta^2 u\left(x,y,t\right)=0 \quad ,$$

where Δ^2 *is the biharmonic operator and* K *is a positive constant. We set* $c=-K^2$ *and write the equation in the form*

$$\frac{\partial^2 u}{\partial t^2}=cu_{xxxx}+2cu_{xxyy}+cu_{yyyy}=\sum_{i=1}^{3}a_i\partial^{\alpha^i}u \quad , \tag{40}$$

where

$$a_1=c \quad , \quad a_2=2c \quad , \quad a_3=c \quad , \quad a=(c,2c,c) \quad ,$$
$$\alpha^1=(4,0) \quad , \quad \alpha^2=(2,2) \quad , \quad \alpha^3=(0,4) \quad .$$

We have $\ell=2$, $\overline{\alpha}=\left(\alpha^1,\alpha^2,\alpha^3\right)$, *and we write* $\sigma=(p,q,r)$, *so that*

$$\overline{\alpha}\cdot\sigma=\alpha^1 p+\alpha^2 q+\alpha^3 r=(4p+2q,2q+4r) \quad ,$$
$$(x,y)^{\overline{\alpha}\cdot\sigma}=x^{4p+2q}y^{4r+2q} \quad , \quad a^\sigma=c^p\left(2c\right)^q c^r=2^q c^{p+q+r} \quad .$$

Setting $\beta=(\beta_1,\beta_2)$, *we find that (29) can be written as*

$$p_{\beta,k}\left(x,t\right)=\beta_1!\beta_2!\sum_{4p+2q\le\beta_1\ ,\ 4r+2q\le\beta_2}\frac{(p+q+r)!}{p!q!r!}\cdot \frac{2^q c^{p+q+r}x^{\beta_1-4p-2q}y^{\beta_2-4r-2q}t^{2p+2q+2r+k}}{\left(\beta_1-4p-2q\right)!\left(\beta_2-4r-2q\right)!\left(2p+2q+2r+k\right)!} \quad .$$

The summation is taken over all multi-indices $\sigma=(p,q,r)$ *such that* $4p+2q\le\beta_1$ *and* $4r+2q\le\beta_2$. *For example, if* $\beta=(6,5)$, *acceptable values of* σ *are*

$$(0,0,0)\ ,\ (0,0,1)\ ,\ (0,1,0)\ ,\ (1,0,0)\ ,\ (1,0,1)\ ,\ (0,2,0)\ ,\ (1,1,0) \quad .$$

After some calculation we find that, with $\beta=(6,5)$ *and* $k=0$, *the polynomial* $p_{\beta,k}\left(x,y,t\right)$ *is*

$$p_{(6,5),0}\left(x,y,t\right)=x^6y^5+60cx^6yt^2+600cx^4y^3t^2 +180cx^2y^5t^2+10800c^2x^2yt^4+2400c^2y^3t^4 \quad .$$

This polynomial solves (40) along with the Cauchy conditions

$$p\left(x,y,0\right)=x^6y^5 \quad , \quad p_t\left(x,y,0\right)=0 \quad .$$

Example 9 *As an example of an equation with a zero order term, consider in one space dimension a special case of the telegraph equation,*

$$\frac{\partial^2 u}{\partial t^2} = Bu + \frac{\partial^2 u}{\partial x^2} \quad .$$

The two multi-indices α *corresponding to space derivatives reduce to scalars,* $\alpha^1 = 0$ *and* $\alpha^2 = 2$*. In (29) also* β *and* γ *are scalars, while* $\sigma = (i, j)$ *is a multi-index in* $\mathbb{R}^2$*. We have* $a = (B, 1)$ *a vector in* $\mathbb{R}^2$*, with* $\overline{\alpha} \cdot \sigma = 2j$ *and* $a^\sigma = B^i$*. Setting* $\ell = 2$*, for* $k = 0$ *and* $k = 1$ *we may write (29) as*

$$p_{\beta,k}(x,t) = \beta! \sum_{i,j:2j\leq\beta} \frac{(i+j)!}{(2i+2j+k)!} \frac{B^i x^{\beta-2j} t^{2i+2j+k}}{(\beta-2j)! i! j!} \quad .$$

The summation is over all multi-indices (i, j) *such that* $2j \leq \beta$*. As there is no restriction on* i*, we may split the sum as*

$$p_{\beta,k}(x,t) = \beta! \sum_{2j\leq\beta} \frac{x^{\beta-2j}}{(\beta-2j)! j!} \sum_{i=0}^{\infty} \frac{(i+j)!}{(2i+2j+k)!} \frac{B^i t^{2i+2j+k}}{i!} \quad , \quad k = 0, 1 \quad .$$

Note that $p_{\beta,k}$ *is a polynomial with respect to* x*, but not with respect to* t*.*

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