SERIES OF POLYNOMIAL SOLUTIONS FOR A CLASS OF EVOLUTION EQUATIONS

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Abstract: We derive pointwise upper bounds on generalized heat polynomials for a class of higher order linear homogeneous evolution equations. These bounds are analogous to those of Rosenbloom and Widder on the heat polynomials, and lead to estimates on the width of the strip of convergence of series expansions in terms of these polynomial solutions. For a subclass of equations including the heat equation, the estimates give the exact width of the strip of convergence. An application is given to a Cauchy problem, wherein the solution is expressed as the sum of a series of polynomial solutions, provided that the Cauchy data is analytic and obeys a growth condition at infinity related to bounds on the coefficients of the differential equation.
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1. Introduction

Let $\mathcal{L}$ be the linear partial differential operator

$$\mathcal{L}u\,(x,t) = \partial_t u\,(x,t) - \sum_{\alpha\in\mathcal{A}} a_\alpha(t)\partial_x^\alpha u\,(x,t) \quad , \tag{1}$$

where $x \in R^n$, $t \in R$, and the coefficients $\{a_\alpha\}$, indexed by a finite collection $\mathcal{A}$ of multi-indices in R^n, are real valued continuous functions of t on an interval

I containing the origin. In [16] we developed explicit formulas for real valued solutions $\{p_\beta\}$ of the initial value problems

$$\mathcal{L}p_\beta(x,t) = 0 \quad , \quad p_\beta(x,0) = x^\beta \quad . \tag{2}$$

For each multi-index β in R^n, p_β is for fixed t a polynomial in x of the form

$$p_\beta(x,t) = \sum_{\nu \leq \beta} c_\nu(t) x^\nu \quad ,$$

with each coefficient c_ν a real valued function in $C^1(I)$. When the coefficients $\{a_\alpha\}$ are constant and $\mathcal{L}$ has no zero order term, p_β is a polynomial in both x and t. The polynomials $\{p_\beta\}$ are analogs of the classical *heat polynomials*, appearing in Appell's work [1], and studied in depth by Rosenbloom and Widder [21, 22, 23, 24]. Indeed, when $\mathcal{L}$ is the heat operator $\partial_t - \Delta$, these polynomials specialize to the heat polynomials.

In [17], following the example of Rosenbloom and Widder for the case of heat polynomials, we studied series expansions

$$u(x,t) = \sum_\beta \frac{c_\beta}{\beta!} p_\beta(x,t) \quad , \tag{3}$$

investigating convergence questions in a horizontal strip $\{(x,t) : |t| < s\}$. We discussed also the Cauchy problem in such a strip,

$$\mathcal{L}u(x,t) = 0 \quad , \quad u(x,0) = f(x) \quad , \tag{4}$$

and gave conditions on f ensuring that a solution can be found as a series (3). In that analysis we derived pointwise upper bounds on the polynomials $\{p_\beta\}$, extending techniques of H. Kemnitz [19], who studied polynomial expansions for solutions of an evolution equation in one space variable,

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^r u(x,t)}{\partial x^r} \quad .$$

The polynomial bounds of [17] lead to conditions on the coefficients $\{c_\beta\}$ ensuring convergence of the series (3).

Whereas the bounds of [17] apply to all operators of the form (1), this broad generality requires compromises limiting the sharpness of the estimates for specific equations. Indeed, one shortcoming is that, when applied to the heat equation, these bounds are weaker than those of Rosenbloom and Widder on the heat polynomials. In this paper we address this situation by studying a smaller class of operators still containing the heat operator, and for these operators we obtain improved upper bounds on the polynomials $\{p_\beta\}$. When applied to the heat operator these bounds are equivalent to those of Rosenbloom and Widder, in the sense that they guarantee the same strip of convergence for series expansions (3). We also consider the Cauchy problem (4) under the assumption that f has a power series expansion

$$f(x) = \sum_{\beta} \frac{c_\beta}{\beta!} x^\beta \quad ,$$

valid in all of R^n. We specify conditions on the coefficients $\{c_\beta\}$ that guarantee convergence of the sum (3) to a solution u of (4) in a horizontal strip of at least a certain width. These conditions at the same time mandate that f can be extended as an entire function to all of C^n, obeying there an exponential growth restriction at infinity. Finally, for a still smaller class of operators, again including the heat operator, we verify that our lower bound on the width of the strip of convergence in fact gives the exact width of this strip.

It is expected that our polynomial estimates, just as in the case of the heat equation, will prove useful in other applications involving the approximation of solutions of $\mathcal{L}u = 0$ by polynomial solutions.

For related work regarding generalizations of the heat polynomials, see [3, 4, 5, 7, 8, 9, 10, 11, 12, 13, 14, 15, 20].

2. The Polynomials

We describe briefly the polynomial solutions $\{p_\beta\}$ of (2), as presented in [16]. First some review of the notation is required. Given $x = (x_1, x_2, \cdots, x_n)$ in R^n and a multi-index $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ in R^n we prescribe

$$\alpha! = \alpha_1!\alpha_2! \cdots \alpha_n! \qquad , \qquad \|\alpha\| = \alpha_1 + \alpha_2 + \cdots + \alpha_n \ , \tag{5}$$

$$x^\alpha = {x_1}^{\alpha_1} {x_2}^{\alpha_2} \cdots {x_n}^{\alpha_n} \qquad , \qquad \partial_x^\alpha = \frac{\partial^{\|\alpha\|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}} \quad . \tag{6}$$

For multi-indices or vectors α and β of the same dimension, we write $\alpha \leq \beta$ whenever $\alpha_i \leq \beta_i$ for all i.

As each a_α appearing in (1) is continuous on an interval I containing the origin, for such a_α we may introduce the antiderivative

$$b_\alpha(t) = \int_0^t a_\alpha(s)\ ds \quad , \ t \in I \ . \tag{7}$$

We let K denote the number of indices α appearing in (1), label these indices as α^1, α^2, $\cdots$, α^K, and label the corresponding coefficients as a_1, a_2, $\cdots$, a_K, so that (1) may be written alternatively as

$$\mathcal{L}u(x,t) = \partial_t u(x,t) - \sum_{k=1}^{K} a_k(t) \partial^{\alpha^k} u(x,t) \quad .$$

We further introduce vector functions

$$a = (a_1, a_2, \cdots, a_K) \quad , \quad b = (b_1, b_2, \cdots, b_K) \quad ,$$

as well as a *vector of multi-indices*

$$\overline{\alpha} = \left(\alpha^1, \alpha^2, \cdots, \alpha^K\right) \quad .$$

Finally, given a multi-index $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_K)$ in R^K, the "dot product" $\overline{\alpha}\cdot\sigma$ is the n-dimensional multi-index

$$\overline{\alpha} \cdot \sigma = \alpha^1\sigma_1 + \alpha^2\sigma_2 + \cdots + \alpha^K\sigma_K \quad .$$

At times it is convenient to write this sum as

$$\overline{\alpha} \cdot \sigma = \sum_{\alpha} \alpha\sigma_\alpha \quad ,$$

where $\sigma_\alpha = \sigma_k$ if $\alpha = \alpha^k$, $1 \leq k \leq K$, and $\sigma_\alpha = 0$ for all other multi-indices α. The polynomials $\{p_\beta\}$ are then prescribed on $R^n \times I$ according to

$$p_\beta(x,t) := \beta! \sum_{\gamma,\sigma \,:\, \gamma+\overline{\alpha}\cdot\sigma=\beta} \frac{x^\gamma b(t)^\sigma}{\gamma!\sigma!} \quad , \tag{8}$$

where the summation is over all multi-indices γ in R^n and σ in R^K having the property that $\gamma + \overline{\alpha} \cdot \sigma = \beta$. An alternative formulation is

$$p_\beta(x,t) := \beta! \sum_{\overline{\alpha}\cdot\sigma\leq\beta} \frac{x^{\beta-\overline{\alpha}\cdot\sigma} b(t)^\sigma}{(\beta - \overline{\alpha}\cdot\sigma)!\sigma!} \quad , \tag{9}$$

where now the sum is over all multi-indices σ in R^K such that $\overline{\alpha} \cdot \sigma \leq \beta$.

If the operator $\mathcal{L}$ of (1) has no zero order term then the summations (8) and (9) extend over only a finite number of terms, and trivially each p_β is as smooth as the individual terms in the sum. Since each b_α is of class C^1 on I, and powers of x are C^∞ on R^n, in this situation each p_β, along with all its partial derivatives involving at most one derivative with respect to t, is continuous in $R^n \times I$.

If the operator $\mathcal{L}$ of (1) has a zero order term, then the summations (8) and (9) involve an infinite number of terms, as now $\alpha = (0, 0, \cdots, 0)$ appears as an index in (1) with corresponding nonzero a_α, resulting in a countably infinite number of multi-indices σ in R^K such that $\overline{\alpha} \cdot \sigma \leq \beta$. However, as shown in [16, 17], in this event the series (8) and (9) are uniformly and absolutely convergent on compact subsets of $R^n \times I$, as are the differentiated series $\partial_t p_\beta$, $\partial_x^\tau p_\beta$, and $\partial_t \partial_x^\tau p_\beta$ for all multi-indices τ in R^n, from which it follows again that all partial derivatives of p_β involving at most one derivative with respect to t, and space derivatives of any order, are continuous in $R^n \times I$.

Because there are only a finite number of powers x^γ with $\gamma \leq \beta$, (8) shows that the functions p_β are polynomials in x for each fixed t. It is demonstrated in [16] also that

$$\partial_x^\alpha p_\beta(x,t) = \begin{cases} \frac{\beta!}{(\beta-\alpha)!}\, p_{\beta-\alpha}(x,t) \quad , & \text{if } \beta \geq \alpha \ , \\ 0 \ , & \text{otherwise,} \end{cases} \tag{10}$$

$$\partial_t p_\beta(x,t) = \beta! \sum_{\alpha\,:\,\alpha\leq\beta} \frac{a_\alpha(t)}{(\beta-\alpha)!}\, p_{\beta-\alpha}(x,t) \quad , \tag{11}$$

with the latter summation over only those α appearing in (1) for which $\alpha \leq \beta$.

Finally, if $\mathcal{L}$ has only constant coefficients $\{a_\alpha\}$, then (7) yields $b_\alpha(t) = t a_\alpha$ for each α, and formulas (8) and (9) simplify to

$$p_\beta(x,t) = \beta! \sum_{\gamma,\sigma\,:\,\gamma+\overline{\alpha}\cdot\sigma=\beta} \frac{a^\sigma x^\gamma t^{\|\sigma\|}}{\gamma!\sigma!} = \beta! \sum_{\overline{\alpha}\cdot\sigma\leq\beta} \frac{a^\sigma x^{\beta-\overline{\alpha}\cdot\sigma} t^{\|\sigma\|}}{(\beta-\overline{\alpha}\cdot\sigma)!\sigma!} \quad .$$

If moreover $\mathcal{L}$ has no zero order term then the sums involve only a finite number of terms, and in this event p_β is a polynomial in both variables (x,t) (as in the case of the heat equation, for example.)

The reader is referred to [16, 17] for further discussion of properties of the functions $\{p_\beta\}$.

3. Factorial Estimates

Recall that the factorial function on the natural numbers can be extended as a real-analytic function to the interval $(-1,\infty)$ by the relation

$$x! = \Gamma(x+1) \quad .$$

From properties of the gamma function it follows that $x!$ is convex and positive on $(-1,\infty)$, approaching $+\infty$ as x approaches either -1 or $+\infty$.

On vectors $x = (x_1, x_2, \cdots, x_n)$ in R^n we use both the Euclidean norm and ℓ^1 norms; respectively, these are

$$|x| = \left({x_1}^2 + {x_2}^2 + \cdots + {x_n}^2\right)^{1/2} \quad , \quad \|x\| = |x_1| + |x_2| + \cdots + |x_n| \quad .$$

For a vector $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ in R^n we write $\alpha \geq 0$ and say α is *nonnegative* whenever $\alpha_i \geq 0$ for each i; for such vectors we may define

$$\alpha! = \alpha_1!\alpha_2!\cdots\alpha_n! \quad ,$$

while the ℓ^1 norm reduces to the right expression of (5). Note that the power x^α of (6) is defined if both $x \geq 0$ and $\alpha \geq 0$, and also for general $x \in R^n$ whenever each α_i is a nonnegative integer.

The next several lemmas summarize some factorial inequalities verified in [17]. (We interpret $0^0 = 1$ in these inequalities.)

Lemma 1 *For real numbers $x, y \geq 0$,*

$$x^x e^{-x} \leq x! \leq x^x e^{1-x}\sqrt{x+1} \quad , \tag{12}$$

$$x!y! \leq (x+y)! \quad . \tag{13}$$

Lemma 2 *Let $x, \alpha \in R^n$, $n \geq 1$, with $\alpha \geq 0$ and $1 < \ell < \infty$. If either (a) each component α_i of α is an integer, or (b) $x \geq 0$, then*

$$\frac{|x^\alpha|}{\alpha!} \leq \frac{1}{(\|\alpha\|/\ell)!} \exp\left[(n\,|x|)^{\ell/(\ell-1)}\right] \quad .$$

Lemma 3 *For $y = (y_1, y_2, \cdots, y_n)$ with all $y_i \geq 0$, and for real numbers $s, r \geq 0$,*

$$\sum_{\sigma \in R^n\ ,\ \|\sigma\| \leq s} \frac{|y^\sigma|\, r^{s-\|\sigma\|}}{\sigma!\,(s-\|\sigma\|)!} \leq \frac{(|y_1| + |y_2| + \cdots + |y_n| + r)^s}{s!} \quad ,$$

where the summation is over all multi-indices $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_n)$ in R^n such that $\|\sigma\| \leq s$.

Lemma 4 *Let $\mathcal{M}$ denote a nonempty collection of M nonzero multi-indices α in R^n, and let m be a positive integer and s a nonnegative real number. Then*

$$\sum_{\|\overline{\alpha}\cdot\sigma\| \leq s} \frac{1}{\left(\frac{s-\|\overline{\alpha}\cdot\sigma\|}{m}\right)! \prod_{\alpha\in\mathcal{M}} \left(\frac{\|\alpha\|}{m}\sigma_\alpha\right)!} \leq m^M\,(M+1)!\frac{(M+1)^{s/m}}{\left(\frac{s}{m}\right)!} \quad ,$$

where the summation is taken over all multi-indices σ in R^M such that

$$\|\overline{\alpha}\cdot\sigma\| = \left\|\sum_{\alpha\in\mathcal{M}} \alpha\sigma_\alpha\right\| = \sum_{\alpha\in\mathcal{M}} \|\alpha\|\,\sigma_\alpha \leq s \quad .$$

4. Polynomial Bounds

We describe conditions under which we derive improved bounds on the polynomials $\{p_\beta\}$.

We assume the highest order of any space derivative in (1) is m, with $m \geq 2$, so that $\|\alpha\| \leq m$ for each $\alpha \in \mathcal{A}$. We also suppose there are finite bounds $\{A_\alpha : \alpha \in \mathcal{A}\}$ such that

$$A_\alpha = \sup_{t \in I} \; |a_\alpha(t)| \quad (\alpha \in \mathcal{A}) \quad . \tag{14}$$

We say that a space derivative ∂_x^α appearing in (1) is *unmixed* if it involves differentiation in at most one space direction – that is, if $\alpha = je_i$ for some nonnegative integer j and integer i, $1 \leq i \leq n$, so that

$$\partial_x^\alpha = \partial_x^{je_i} = \frac{\partial^j}{\partial {x_i}^j} \quad .$$

We say also that $\alpha = je_i$ is an *unmixed* multi-index. All other space derivatives and corresponding $\alpha's$ are *mixed.* We list the collection of unmixed multi-indices α in R^n of magnitude m according to

$$\alpha^1 = me_1 \quad , \quad \alpha^2 = me_2 \quad , \quad \cdots \quad , \quad \alpha^n = me_n \quad , \tag{15}$$

and the coefficients corresponding to these as

$$a_i = a_{\alpha^i} = a_{me_i} \quad , \; 1 \leq i \leq n \quad .$$

We assume that all unmixed derivatives of order m are present in (1), and accordingly we may write this operator as

$$\begin{aligned} \mathcal{L}u\,(x,t) \quad &= \quad \partial_t u\,(x,t) - \sum_{i=1}^n a_i(t) \frac{\partial^m u\,(x,t)}{\partial {x_i}^m} \qquad (16) \\ &- \sum_{\alpha \in \mathcal{A},\alpha \text{ mixed},\|\alpha\|=m} a_\alpha(t) \partial_x^\alpha u\,(x,t) - \sum_{\alpha \in \mathcal{A},\|\alpha\|<m} a_\alpha(t) \partial_x^\alpha u\,(x,t) \quad . \end{aligned}$$

Regarding further the coefficients of the m-th order unmixed derivatives, we presume for $1 \leq i \leq n$ the lower and upper bounds

$$0 < \varepsilon_i := \inf_{t \in I} \; |a_i(t)| \leq \sup_{t \in I} \; |a_i(t)| =: A_i \quad . \tag{17}$$

(Thus, $A_i = A_{me_i}$ in the notation of (14).) Finally, A and ε will denote vectors with these bounds as entries; that is,

$$A := (A_1, A_2, \cdots, A_n) \qquad , \qquad \varepsilon := (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n) \quad . \tag{18}$$

Theorem 5 *Assume $m \geq 2$, where m is the order of (16), and that the coefficents $\{a_\alpha\}$ are continuous with the bounds (14) and (17) on an interval I containing the origin. Then for $x \in R^n$ and $t \in I$, and for any $\delta > 0$,*

$$\frac{|p_\beta(x,t)|}{\beta!} \leq \frac{m^M\,(M+1)!A^{\beta/m}}{(\beta/m)!} \cdot \tag{19}$$
$$\cdot \left(|t| + |t| \sum_{\alpha\in\mathcal{A}\ ,\ \alpha \text{ mixed}\ ,\ \|\alpha\|=m} \frac{A_\alpha}{\varepsilon^{\alpha/m}} + \delta \right)^{\|\beta\|/m} \cdot$$
$$\cdot \exp\left[\left(\frac{M+1}{\delta}\right)^{1/(m-1)} \left(n \left\| \left(\frac{x_1}{\varepsilon_1^{1/m}}, \cdots, \frac{x_n}{\varepsilon_n^{1/m}} \right) \right\| \right)^{m/(m-1)} \right.$$
$$\left. + \sum_{\alpha\in\mathcal{A}\ ,\ \|\alpha\|<m} \left(\frac{M+1}{\delta}\right)^{\|\alpha\|/(m-\|\alpha\|)} \left(\frac{A_\alpha\,|t|}{\varepsilon^{\alpha/m}}\right)^{m/(m-\|\alpha\|)} \right] ,$$

where M is the number of multi-indices $\alpha \in \mathcal{A}$ such that $0 < \|\alpha\| < m$, and A and ε are the vectors (18).

Proof: Initially we assume (16) contains no zero order term; that is, there is no term in the sum corresponding to $\alpha = (0, \cdots, 0)$. Let K denote the number of space derivative terms in this sum; then by (8),

$$\frac{p_\beta(x,t)}{\beta!} = \sum_{\gamma+\overline{\alpha}\cdot\sigma=\beta} \frac{x^\gamma b(t)^\sigma}{\gamma!\sigma!} \quad , \tag{20}$$

where the sum is over all multi-indices γ in R^n and σ in R^K satisfying $\gamma+\overline{\alpha}\cdot\sigma = \beta$. We let

$$\rho(t) = (\rho_1(t), \rho_2(t), \cdots, \rho_n(t)) \tag{21}$$

denote a vector of positive functions of t on I, to be specified later. (Henceforth, for brevity we suppress the arguments of functions of t.) Given $\delta > 0$, Lemma 2 (with $\ell = m$ and $\alpha = \gamma$) yields

$$\frac{|x^\gamma|}{\gamma!} = \frac{\rho^\gamma \delta^{\|\gamma\|/m}}{\gamma!} \left| \left(\frac{x_1}{\rho_1\delta^{1/m}}, \frac{x_2}{\rho_2\delta^{1/m}}, \cdots, \frac{x_n}{\rho_n\delta^{1/m}} \right)^\gamma \right|$$
$$\leq \frac{\rho^\gamma \delta^{\|\gamma\|/m}}{(\|\gamma\|/m)!} \exp\left[\left| \frac{n}{\delta^{1/m}} \left(\frac{x_1}{\rho_1}, \frac{x_2}{\rho_2}, \cdots, \frac{x_n}{\rho_n} \right) \right|^{m/(m-1)} \right] .$$

This estimate, used in (20), leads to

$$\frac{|p_\beta(x,t)|}{\beta!} \leq \exp\left[\left| \frac{n}{\delta^{1/m}} \left(\frac{x_1}{\rho_1}, \frac{x_2}{\rho_2}, \cdots, \frac{x_n}{\rho_n} \right) \right|^{m/(m-1)} \right] \cdot S \quad , \tag{22}$$

where

$$S = \sum_{\gamma+\overline{\alpha}\cdot\sigma=\beta} \frac{\rho^{\gamma}\delta^{\|\gamma\|/m}}{(\|\gamma\|/m)!} \frac{|b^{\sigma}|}{\sigma!} \quad . \tag{23}$$

We let N denote the number of terms in the summation of (16) corresponding to $\|\alpha\| = m$, and M the number with $0 < \|\alpha\| < m$, so that $K = N + M$. We number the multi-indices appearing in this sum as

$$\alpha^1 \ , \ \alpha^2 \ , \ \cdots \ , \ \alpha^N \ , \ \alpha^{N+1} \ , \ \alpha^{N+2} \ , \ \cdots \ , \ \alpha^{N+M} \ ,$$

where $\alpha^1, \cdots, \alpha^N$ are the highest order multi-indices with $\|\alpha\| = m$, and $\alpha^{N+1}, \cdots, \alpha^{N+M}$ are of lower order. Thus, in view of (15), $N \geq n$. We write a general multi-index σ in R^K as

$$\sigma = (\nu_1, \cdots, \nu_N, \lambda_1, \cdots, \lambda_M) = \nu + \lambda \ ,$$

where the multi-indices ν and λ, each also in R^K, are defined as

$$\nu = (\nu_1, \cdots, \nu_N, 0, \cdots, 0) \quad , \quad \lambda = (0, \cdots, 0, \lambda_1, \cdots, \lambda_M) \quad , \tag{24}$$

and

$$\overline{\alpha} \cdot \sigma = \sum \alpha \sigma_\alpha = \sum_{i=1}^{N} \alpha^i \nu_i + \sum_{i=1}^{M} \alpha^{N+i} \lambda_i = \overline{\alpha} \cdot \nu + \overline{\alpha} \cdot \lambda \ .$$

(We admit the possibility $M = 0$, in which case $K = N$, $\lambda = (0, \cdots, 0)$ and $\sigma = \nu$.) Then $\sigma! = \nu!\lambda!$, and (23) can be written as

$$\begin{aligned} S &= \sum_{\gamma+\overline{\alpha}\cdot\nu+\overline{\alpha}\cdot\lambda=\beta} \frac{\rho^{\gamma}\delta^{\|\gamma\|/m}}{(\|\gamma\|/m)!} \frac{|b^{\nu}|}{\nu!} \frac{\left|b^{\lambda}\right|}{\lambda!} \\ &= \sum_{\overline{\alpha}\cdot\nu+\overline{\alpha}\cdot\lambda\leq\beta} \frac{\rho^{\beta-\overline{\alpha}\cdot\nu-\overline{\alpha}\cdot\lambda}\delta^{\|\beta-\overline{\alpha}\cdot\nu-\overline{\alpha}\cdot\lambda\|/m}}{(\|\beta-\overline{\alpha}\cdot\nu-\overline{\alpha}\cdot\lambda\|/m)!} \frac{|b^{\nu}|}{\nu!} \frac{\left|b^{\lambda}\right|}{\lambda!} \\ &= \sum_{\nu\,:\,\overline{\alpha}\cdot\nu\leq\beta} \rho^{\beta-\overline{\alpha}\cdot\nu}\delta^{\|\beta-\overline{\alpha}\cdot\nu\|/m} \frac{|b^{\nu}|}{\nu!} \cdot T \quad , \end{aligned} \tag{25}$$

where

$$T = \frac{1}{(\|\beta-\overline{\alpha}\cdot\nu\|/m)!} \quad , \text{ if } M = 0 \ , \tag{26}$$

$$T = \sum_{\lambda\,:\,\overline{\alpha}\cdot\lambda\leq\beta-\overline{\alpha}\cdot\nu} \frac{\rho^{-\overline{\alpha}\cdot\lambda}\delta^{-\|\overline{\alpha}\cdot\lambda\|/m}}{(\|\beta-\overline{\alpha}\cdot\nu-\overline{\alpha}\cdot\lambda\|/m)!} \frac{\left|b^{\lambda}\right|}{\lambda!} \quad , \text{ if } M > 0 \ . \tag{27}$$

In the case $M > 0$ we let $\mathcal{M}$ denote the collection of multi-indices α appearing in (16) such that $0 < \|\alpha\| < m$, and apply Lemma 2 but with

$\ell = m/\left\|\alpha\right\|$ and $n = 1$ to estimate

$$\begin{aligned}
\frac{\left|b^{\lambda}\right|}{\lambda!} &= \prod_{\alpha\in\mathcal{M}} \frac{\left|b_\alpha\right|^{\lambda_\alpha}}{\lambda_\alpha!} = \prod_{\alpha\in\mathcal{M}} \left\{ \rho^{\alpha\lambda_\alpha}\delta^{\|\alpha\|\lambda_\alpha/m} \frac{1}{\lambda_\alpha!}\left(\frac{|b_\alpha|}{\rho^{\alpha}\delta^{\|\alpha\|/m}}\right)^{\lambda_\alpha}\right\} \\
&\le \prod_{\alpha\in\mathcal{M}} \left\{ \rho^{\alpha\lambda_\alpha}\delta^{\|\alpha\|\lambda_\alpha/m} \frac{1}{\left(\frac{\|\alpha\|\lambda_\alpha}{m}\right)!} \exp\left[\left(\frac{|b_\alpha|}{\rho^{\alpha}\delta^{\|\alpha\|/m}}\right)^{m/(m-\|\alpha\|)}\right]\right\} \\
&= \rho^{\overline{\alpha}\cdot\lambda}\delta^{\|\overline{\alpha}\cdot\lambda\|/m} \exp\left[\sum_{\alpha\in\mathcal{M}}\left(\frac{|b_\alpha|}{\rho^{\alpha}\delta^{\|\alpha\|/m}}\right)^{m/(m-\|\alpha\|)}\right] \prod_{\alpha\in\mathcal{M}} \frac{1}{\left(\frac{\|\alpha\|\lambda_\alpha}{m}\right)!} \quad .
\end{aligned}$$

We substitute this inequality into (27), noting that $\|\overline{\alpha}\cdot\lambda\| \le \|\beta - \overline{\alpha}\cdot\nu\|$ in this sum, to obtain

$$\begin{aligned}
T &\le \exp\left[\sum_{\alpha\in\mathcal{M}}\left(\frac{|b_\alpha|}{\rho^{\alpha}\delta^{\|\alpha\|/m}}\right)^{m/(m-\|\alpha\|)}\right] \\
&\cdot \sum_{\lambda\,:\,\|\overline{\alpha}\cdot\lambda\|\le\|\beta-\overline{\alpha}\cdot\nu\|} \frac{1}{\left(\frac{\|\beta-\overline{\alpha}\cdot\nu\|-\|\overline{\alpha}\cdot\lambda\|}{m}\right)!\prod_{\alpha\in\mathcal{M}}\left(\frac{\|\alpha\|}{m}\lambda_\alpha\right)!} \quad .
\end{aligned}$$

In the latter sum note that $\overline{\alpha}\cdot\lambda = \sum_{\alpha\in\mathcal{M}}\alpha\lambda_\alpha$; if we ignore the leading zeros of λ and view the remainder of λ as a multi-index in R^M, we may apply Lemma 4 to this sum to arrive at

$$T \le m^M (M+1)! \frac{(M+1)^{\|\beta-\overline{\alpha}\cdot\nu\|/m}}{\left(\frac{\|\beta-\overline{\alpha}\cdot\nu\|}{m}\right)!} \exp\left[\sum_{\alpha\in\mathcal{M}}\left(\frac{|b_\alpha|}{\rho^{\alpha}\delta^{\|\alpha\|/m}}\right)^{m/(m-\|\alpha\|)}\right] \quad .$$

Observe that the right side reduces to that of (26) in the case $M = 0$, when $\mathcal{M}$ is empty, so that in fact this last estimate is valid for both $M > 0$ and $M = 0$. We insert this inequality into (25), and then (25) into (22), to obtain

$$\begin{aligned}
&\tfrac{\left|p_\beta(x,t)\right|}{\beta!} \le m^M (M+1)! \cdot U \qquad (28)\\
&\cdot \exp\left[\left|\tfrac{n}{\delta^{1/m}}\left(\tfrac{x_1}{\rho_1},\tfrac{x_2}{\rho_2},\cdots,\tfrac{x_n}{\rho_n}\right)\right|^{m/(m-1)} + \textstyle\sum_{\alpha\in\mathcal{M}}\left(\tfrac{|b_\alpha|}{\rho^{\alpha}\delta^{\|\alpha\|/m}}\right)^{m/(m-\|\alpha\|)}\right] \ ,
\end{aligned}$$

where

$$U = \sum_{\nu\,:\,\overline{\alpha}\cdot\nu\le\beta} \frac{|b^\nu|}{\nu!} \frac{\rho^{\beta-\overline{\alpha}\cdot\nu}\delta^{\|\beta-\overline{\alpha}\cdot\nu\|/m}(M+1)^{\|\beta-\overline{\alpha}\cdot\nu\|/m}}{(\|\beta-\overline{\alpha}\cdot\nu\|/m)!} \quad . \qquad (29)$$

As in (15) we let α^1, $\cdots$, α^n denote the *unmixed* multi-indices in $\mathcal{A}$ of highest order m, and we continue to list the *mixed* multi-indices in $\mathcal{A}$ of order m as

$$\alpha^{n+1} \quad , \quad \alpha^{n+2} \quad , \quad \cdots \quad , \quad \alpha^{n+P} \quad ,$$

so that $N = n + P$. Then we can write a multi-index ν, as specified in (24), as

$$\nu = (\tau_1, \cdots, \tau_n, \mu_1, \cdots, \mu_P, 0, \cdots, 0) = \tau + \mu \quad ,$$

where τ and μ are multi-indices in R^K prescribed by

$$\tau = (\tau_1, \cdots, \tau_n, 0, \cdots, 0) \quad , \quad \mu = (0, \cdots, 0, \mu_1, \cdots, \mu_P, 0, \cdots, 0) \quad .$$

(We allow the case when there are no mixed multi-indices of order m, in which event $P = 0$, $\mu = (0, \cdots, 0)$, and $\nu = \tau$.) Then

$$\overline{\alpha} \cdot \nu = \overline{\alpha} \cdot \tau + \overline{\alpha} \cdot \mu \quad , \quad \|\overline{\alpha} \cdot \tau\| = m \|\tau\| \quad , \quad \|\overline{\alpha} \cdot \mu\| = m \|\mu\| \quad ,$$

and we may write (29) as

$$\begin{aligned} U &= \sum_{\overline{\alpha}\cdot\tau + \overline{\alpha}\cdot\mu \leq \beta} \frac{|b^{\tau+\mu}|}{\tau!\mu!} \frac{\rho^{\beta - \overline{\alpha}\cdot\tau - \overline{\alpha}\cdot\mu} \left[\delta (M+1)\right]^{\|\beta\|/m - \|\tau\| - \|\mu\|}}{(\|\beta\|/m - \|\tau\| - \|\mu\|)!} \qquad (30) \\ &= \rho^\beta \sum_{\tau \,:\, \overline{\alpha}\cdot\tau \leq \beta} \frac{|b^\tau| \rho^{-\overline{\alpha}\cdot\tau}}{\tau!} \cdot V \quad , \end{aligned}$$

where

$$\begin{aligned} V &= \frac{\left[\delta (M+1)\right]^{\|\beta\|/m - \|\tau\|}}{(\|\beta\|/m - \|\tau\|)!} \quad , \text{ if } P = 0 \quad , \qquad (31) \\ V &= \sum_{\mu \,:\, \overline{\alpha}\cdot\mu \leq \beta - \overline{\alpha}\cdot\tau} \frac{|b^\mu| \rho^{-\overline{\alpha}\cdot\mu} \left[\delta (M+1)\right]^{\|\beta\|/m - \|\tau\| - \|\mu\|}}{\mu! (\|\beta\|/m - \|\tau\| - \|\mu\|)!} \quad , \text{ if } P > 0 \ . \end{aligned}$$

When $P > 0$ we note that, in the summation for V,

$$\begin{aligned} \overline{\alpha} \cdot \mu \leq \beta - \overline{\alpha} \cdot \tau &\Longrightarrow m \|\mu\| \leq \|\beta\| - m \|\tau\| \quad , \\ |b^\mu| \rho^{-\overline{\alpha}\cdot\mu} &= \prod_{\alpha \text{ mixed} \,,\, \|\alpha\| = m} \left(\frac{|b_\alpha|}{\rho^\alpha} \right)^{\mu_\alpha} \quad ; \end{aligned}$$

then with the help of Lemma 3 we deduce that

$$\begin{aligned} V &\leq \sum_{\|\mu\| \leq \|\beta\|/m - \|\tau\|} \frac{1}{\mu!} \left\{ \prod_{\alpha \text{ mixed} \,,\, \|\alpha\| = m} \left(\frac{|b_\alpha|}{\rho^\alpha} \right)^{\mu_\alpha} \right\} \frac{\left[\delta (M+1)\right]^{\|\beta\|/m - \|\tau\| - \|\mu\|}}{(\|\beta\|/m - \|\tau\| - \|\mu\|)!} \\ &\leq \frac{1}{(\|\beta\|/m - \|\tau\|)!} \left[\sum_{\alpha \text{ mixed} \,,\, \|\alpha\| = m} \frac{|b_\alpha|}{\rho^\alpha} + \delta (M+1) \right]^{\|\beta\|/m - \|\tau\|} \quad . \end{aligned}$$

As the right side reduces to that of (31) when $P = 0$, this inequality holds likewise in that case. Now setting

$$Z = \sum_{\alpha \text{ mixed}\ ,\ \|\alpha\|=m} \frac{|b_\alpha|}{\rho^\alpha} + \delta (M+1) \quad ,$$

we substitute the last inequality for V into (30) to produce

$$U \leq \rho^\beta \sum_{\tau\,:\,\overline{\alpha}\cdot\tau \leq \beta} \frac{|b^\tau|\, \rho^{-\overline{\alpha}\cdot\tau}}{\tau!} \frac{Z^{\|\beta\|/m - \|\tau\|}}{(\|\beta\|/m - \|\tau\|)!} \quad . \tag{32}$$

At this point we make our choice of $\rho(t)$ in (21), setting

$$\rho_i(t) := \left(\frac{|b_i(t)|}{|t|}\right)^{1/m} \quad , \quad 1 \leq i \leq n \quad , \tag{33}$$

where at $t = 0$ we use, in accordance with (7),

$$b_i(t) = \int_0^t a_i(s)\, ds \quad , \tag{34}$$

to interpret $\rho_i(0)$ as $\lim_{t\to 0} \rho_i(t) = |a_i(0)|^{1/m}$. Observe that $\rho_i(t) > 0$ by (34) and (17). Then in (32) we have

$$\begin{aligned}
\overline{\alpha}\cdot\tau &= \sum_{i=1}^n (m e_i)\, \tau_i = \sum_{i=1}^n (m\tau_i)\, e_i \quad , \\
|b^\tau|\, \rho^{-\overline{\alpha}\cdot\tau} &= \prod_{i=1}^n \frac{|b_i|^{\tau_i}}{(\rho_i)^{m\tau_i}} = \prod_{i=1}^n |t|^{\tau_i} = |t|^{\|\tau\|} \quad ,
\end{aligned}$$

while (13) implies that

$$(\|\beta\|/m - \|\tau\|)! = \left[\sum_{i=1}^n \left(\frac{\beta_i}{m} - \tau_i\right)\right]! \geq \prod_{i=1}^n \left(\frac{\beta_i}{m} - \tau_i\right)! \quad .$$

Consequently, (32) leads to

$$U \leq \rho^\beta \sum_{\tau_1 \leq \beta_1/m} \frac{|t|^{\tau_1} Z^{\beta_1/m - \tau_1}}{\tau_1!\, (\beta_1/m - \tau_1)!} \sum_{\tau_2 \leq \beta_2/m} \cdots \sum_{\tau_n \leq \beta_n/m} \frac{|t|^{\tau_n} Z^{\beta_n/m - \tau_n}}{\tau_n!\, (\beta_n/m - \tau_n)!} \quad ,$$

which with n applications of the scalar version of Lemma 3 yields

$$\begin{aligned}
U &\leq \rho^\beta \frac{1}{(\beta_1/m)!} \left[|t| + Z\right]^{\beta_1/m} \cdots \frac{1}{(\beta_n/m)!} \left[|t| + Z\right]^{\beta_n/m} \\
&= \frac{\rho(t)^\beta}{(\beta/m)!} \left(|t| + \sum_{\alpha \text{ mixed}\ ,\ \|\alpha\|=m} \frac{|b_\alpha(t)|}{\rho(t)^\alpha} + \delta (M+1)\right)^{\|\beta\|/m} \quad .
\end{aligned}$$

The bounds (14) and (17), applied to (7), (33) and (34), imply

$$\begin{array}{ccc} |b_\alpha(t)| \le A_\alpha\,|t| & , & \varepsilon_i\,|t| \le |b_i(t)| \le A_i\,|t| \ , \\ \varepsilon_i^{1/m} \le \rho_i(t) \le A_i^{1/m} & , & \varepsilon^{\alpha/m} \le \rho\,(t)^\alpha \le A^{\alpha/m} \ , \end{array}$$

which with the previous inequality gives

$$U \le \frac{A^{\beta/m}}{(\beta/m)!}\left(|t| + |t| \sum_{\alpha \text{ mixed },\ \|\alpha\|=m} \frac{A_\alpha}{\varepsilon^{\alpha/m}} + \delta\,(M+1)\right)^{\|\beta\|/m} \quad .$$

We substitute these last few inequalities into (28), replace δ by $\delta/\,(M+1)$, and obtain the desired inequality (19).

It remains now only to verify the theorem when the operator (16) has a zero order term, corresponding to $\alpha = (0,\cdots,0)$. To analyze this situation, we assume the operator $\mathcal{L}$ of (16) has no zero order term, and we subtract such a term from $\mathcal{L}$ to produce a modified operator

$$\begin{aligned} \mathcal{L}_0 u\,(x,t) &= \partial_t u\,(x,t) - \sum_{\alpha\in\mathcal{A}} a_\alpha(t)\partial_x^\alpha u\,(x,t) - a_0(t)u\,(x,t) \\ &= \partial_t u\,(x,t) - \sum_{\alpha\in\mathcal{B}} a_\alpha(t)\partial_x^\alpha u\,(x,t) \quad , \end{aligned}$$

where $\mathcal{B} = \mathcal{A} \cup \{(0,\cdots,0)\}$. We let $\{q_\beta\}$ denote the polynomials (8) for $\mathcal{L}_0$, and $\{p_\beta\}$ the corresponding polynomials for $\mathcal{L}$. The analogue of (20) for q_β is

$$\frac{q_\beta\,(x,t)}{\beta!} = \sum_{\gamma+\overline{\alpha}\cdot\sigma=\beta} \frac{x^\gamma b(t)^\sigma}{\gamma!\sigma!} \quad , \tag{35}$$

where here

$$\overline{\alpha}\cdot\sigma = \sum_{\alpha\in\mathcal{B}} \alpha\sigma_\alpha = \sum_{\alpha\in\mathcal{A}} \alpha\sigma_\alpha + (0,\cdots,0)\,\sigma_0 = \sum_{\alpha\in\mathcal{A}} \alpha\sigma_\alpha \quad .$$

Thus the condition $\gamma + \overline{\alpha}\cdot\sigma = \beta$ places no restriction on the component σ_0 of σ corresponding to $\alpha = (0,\cdots,0)$; that is, for fixed acceptable choices of the other components $\{\sigma_\alpha\}$ of σ, we may allow σ_0 to range over all nonnegative integers. If we write

$$\sigma = (\xi,\sigma_0) = (\xi_1,\cdots,\xi_K,\sigma_0) \quad , \quad b_0(t) = \int_0^t a_0(s)\,ds \ ,$$

then (35) may be written as

$$\frac{q_\beta\,(x,t)}{\beta!} = \sum_{\gamma+\overline{\alpha}\cdot\xi=\beta} \frac{x^\gamma b(t)^\xi}{\gamma!\xi!} \sum_{\sigma_0=0}^{\infty} \frac{b_0(t)^{\sigma_0}}{(\sigma_0)!} \ , \tag{36}$$

where here we interpret

$$\overline{\alpha}\cdot\xi = \sum_{\alpha\in\mathcal{A}} \alpha\xi_\alpha \quad , \quad b(t)^\xi = \prod_{\alpha\in\mathcal{A}} b_\alpha(t)^{\xi_\alpha} \quad .$$

Thus equation (36) confirms the formula

$$q_\beta(x,t) = p_\beta\,(x,t)\ e^{b_0(t)} \quad .$$

But $b_0(t) \le A_0\,|t|$ where $|a_0(t)| \le A_0$, giving

$$|q_\beta\,(x,t)| \le e^{A_0|t|}\,|p_\beta\,(x,t)| \quad .$$

As p_β has in Theorem 5 the bound (19), the identical bound holds for q_β because the multiplicative factor $e^{A_0|t|}$ only adds another term, corresponding to $\alpha = (0,\cdots,0)$, to the sum in the expression

$$\exp\left[\sum_{\alpha\in\mathcal{A}\ ,\ \|\alpha\|<m} \left(\frac{M+1}{\delta}\right)^{\|\alpha\|/(m-\|\alpha\|)} |A_\alpha t|^{m/(m-\|\alpha\|)}\right] \quad ,$$

so that $\alpha\in\mathcal{A}$ changes to $\alpha\in\mathcal{B}$. The number M of multi-indices α satisfying $0<\|\alpha\|<m$ is the same for $\mathcal{A}$ and $\mathcal{B}$. These remarks complete the proof of the theorem. $\diamondsuit$

When the operator (16) has no lower order terms, assuming the form

$$\mathcal{L}u\,(x,t) = \partial_t u\,(x,t) - \sum_{\|\alpha\|=m} a_\alpha(t)\partial_x^\alpha u\,(x,t) \quad ,$$

the bounds on the polynomials $\{p_\beta\}$ are simplified; we have $M=0$, while sums over lower order multi-indices are 0. Inequality (19) reduces to

$$\begin{aligned} \frac{|p_\beta(x,t)|}{\beta!} \quad &\le \quad \frac{A^{\beta/m}}{(\beta/m)!}\left(|t| + |t| \sum_{\alpha \text{ mixed}\ ,\ \|\alpha\|=m} \frac{A_\alpha}{\varepsilon^{\alpha/m}} + \delta\right)^{\|\beta\|/m} \\ &\cdot \quad \exp\left[\delta^{-1/(m-1)}\left(n\left|\left(\frac{x_1}{\varepsilon_1^{1/m}},\cdots,\frac{x_n}{\varepsilon_n^{1/m}}\right)\right|\right)^{m/(m-1)}\right] \quad . \end{aligned} \tag{37}$$

Of special interest is the heat operator in n space dimensions,

$$\mathcal{H}u\,(x,t) = \partial_t u\,(x,t) - \Delta u\,(x,t) = \partial_t u(x,t) - \sum_{i=1}^n \frac{\partial^2 u\,(x,t)}{\partial {x_i}^2} \quad .$$

For this equation $A_\alpha = 1$ if α is unmixed and $A_\alpha = 0$ if α is mixed, while $m=2$ and $\varepsilon_i = 1$ for each i; accordingly (37) becomes

$$\frac{|p_\beta(x,t)|}{\beta!} \le \frac{(|t|+\delta)^{\|\beta\|/2}}{(\beta/2)!}\exp\left[\frac{(n\,|x|)^2}{\delta}\right] \quad . \tag{38}$$

This estimate is analogous to bounds of Rosenbloom and Widder [21, 22, 23, 24], although not directly comparable as their bounds are stated somewhat differently, depending on the cases $t \geq 0$ and $t < 0$. We will see later, however, that (38) and the bounds of Rosenbloom and Widder give the same strips of convergence for series of the form (3).

5. Series Expansions

We consider polynomial series of the form

$$u(x,t) = \sum_{\beta} \frac{c_\beta}{\beta!}\, p_\beta\,(x,t) \quad , \tag{39}$$

where again the polynomials $\{p_\beta\}$ are those associated with the operator (1). Expanding on techniques of [19, 21, 22, 23, 24], we prescribe conditions on the coefficients $\{c_\beta\}$ that guarantee convergence of the series in a strip

$$S_s = \{(x,t) : x \in R^n,\ t \in I,\ |t| < s\} \quad .$$

Given a real valued function f on multi-indices β in R^n, we define

$$\overline{\lim}_{\beta\to\infty}\, f\,(\beta) = \lim\sup_{\beta\to\infty}\, f\,(\beta) := \lim_{k\to\infty}\ \sup_{\|\beta\|\geq k}\, f\,(\beta) \quad .$$

Theorem 6 *Under the hypotheses of Theorem 5, assume that*

$$\overline{\lim}_{\beta\to\infty}\, \frac{|c_\beta|^{m/\|\beta\|}\, meA^{\beta/\|\beta\|}}{\beta^{\beta/\|\beta\|}} \left(1 + \sum_{\alpha\in\mathcal{A},\alpha\ \text{mixed},\|\alpha\|=m} \frac{A_\alpha}{\varepsilon^{\alpha/m}}\right) = \frac{1}{s} \quad , \tag{40}$$

where $0 < s \leq \infty$. *Then*

(a) the series (39) converges absolutely and uniformly on compact subsets of the strip S_s *to a function* u.

(b) both u *and* $\partial u/\partial t$ *are continuous in this strip, as are their space derivatives* $\partial_x^\nu u$ *and* $\partial_x^\nu u_t$ *of all orders. All these derivatives can be obtained from (39) by termwise differentiation, with convergence of the differentiated series likewise absolute and uniform on compact subsets of* S_s. *Finally,* u *solves in* S_s *the equation* $\mathcal{L}u = 0$, *where* $\mathcal{L}$ *is the operator (1), while for each* β,

$$c_\beta = \partial_x^\beta u(0,0) \quad . \tag{41}$$

Proof: Assume the hypotheses of Theorem 5 hold. Given X, T with $0 < X < \infty$, $0 < T < s$, consider the compact region $R(X,T)$ containing all points (x,t) such that

$$(x,t) \in R^n \times I \quad , \quad |x| \leq X \quad , \quad |t| \leq T \quad .$$

Given $\delta > 0$, inequality (19) guarantees a constant C, depending on m, n, M, X, T, δ, $\{A_\alpha\}$, and ε, but *independent* of β, such that for all $(x,t) \in R(X,T)$,

$$\frac{|p_\beta(x,t)|}{\beta!} \leq C \, \frac{A^{\beta/m} (LT+\delta)^{\|\beta\|/m}}{(\beta/m)!} \quad , \tag{42}$$

where

$$L := 1 + \sum_{\alpha \in \mathcal{A} \ , \ \alpha \text{ mixed } , \ \|\alpha\| = m} \frac{A_\alpha}{\varepsilon^{\alpha/m}} \quad .$$

Applying the left inequality of (12), in (42) we have

$$\frac{1}{(\beta/m)!} = \prod_{i=1}^{n} \frac{1}{(\beta_i/m)!} \leq \prod_{i=1}^{n} \left(\frac{em}{\beta_i}\right)^{\beta_i/m} = \frac{(em)^{\|\beta\|/m}}{\beta^{\beta/m}} \quad . \tag{43}$$

Let T_1 satisfy $T < T_1 < s$. Under assumption (40) there exists $k > 0$ such that $\|\beta\| \geq k$ implies

$$\frac{|c_\beta|^{m/\|\beta\|} A^{\beta/\|\beta\|} meL}{\beta^{\beta/\|\beta\|}} \leq \frac{1}{T_1} \quad . \tag{44}$$

We combine (42), (43), and (44) to infer that, for $\|\beta\| \geq k$,

$$|c_\beta| \, \frac{|p_\beta(x,t)|}{\beta!} \leq C \left(\frac{LT+\delta}{LT_1}\right)^{\|\beta\|/m} \quad . \tag{45}$$

As L is positive and $T_1 > T$, in (45) we can choose δ sufficiently small that the quanitity in parentheses on the right is less than some positive constant r, with $r < 1$; then

$$\sum_{\|\beta\| \geq k} \left| \frac{c_\beta}{\beta!} \, p_\beta(x,t) \right| \leq C \sum_{\|\beta\| \geq k} r^{\|\beta\|/m} < \infty \quad .$$

Thus (39) converges absolutely and uniformly in the region $R(X,T)$, and consequently also on compact subsets of S_s because X can be arbitrarily large and T arbitrarily close to s.

Keeping in mind the identity (10), we consider a formal space derivative of the series (39),

$$\begin{aligned} \partial_x^\nu u(x,t) &= \partial_x^\nu \sum_\beta \frac{c_\beta}{\beta!} \, p_\beta(x,t) = \sum_\beta \frac{c_\beta}{\beta!} \, \partial_x^\nu p_\beta(x,t) \\ &= \sum_{\beta \geq \nu} \frac{c_\beta}{(\beta-\nu)!} \, p_{\beta-\nu}(x,t) = \sum_\beta \frac{c_{\beta+\nu}}{\beta!} \, p_\beta(x,t) \quad . \end{aligned} \tag{46}$$

Under assumption (40), for the coefficients in the derived series we find that

$$\begin{aligned}
&\overline{\lim}_{\beta\to\infty} \frac{\left|c_{\beta+\nu}\right|^{m/\|\beta\|} A^{\beta/\|\beta\|} meL}{\beta^{\beta/\|\beta\|}} \\
&= \overline{\lim}_{\beta\to\infty} \left\{ \left[\frac{\left|c_{\beta+\nu}\right|^{m/\|\beta+\nu\|} A^{(\beta+\nu)/\|\beta+\nu\|} meL}{(\beta+\nu)^{(\beta+\nu)/\|\beta+\nu\|}} \right]^{\|\beta+\nu\|/\|\beta\|} \right. \\
&\qquad \left. \cdot \frac{(\beta+\nu)^{\nu/\|\beta\|}}{A^{\nu/\|\beta\|}} \frac{1}{(meL)^{\|\nu\|/\|\beta\|}} \frac{(\beta+\nu)^{\beta/\|\beta\|}}{\beta^{\beta/\|\beta\|}} \right\} \quad .
\end{aligned}$$

As $\beta \to \infty$ we have $A^{\nu/\|\beta\|} \to 1$, $(meL)^{\|\nu\|/\|\beta\|} \to 1$, and

$$1 \leq (\beta+\nu)^{\nu/\|\beta\|} \leq (\|\beta\| + \|\nu\|)^{\|\nu\|/\|\beta\|} \to 1 \quad ,$$

$$1 \leq \frac{(\beta+\nu)^{\beta/\|\beta\|}}{\beta^{\beta/\|\beta\|}} = \left[\prod_{i=1}^{n} \frac{(\beta_i+\nu_i)^{\beta_i}}{\beta_i^{\beta_i}} \right]^{1/\|\beta\|} \leq \left[\prod_{i=1}^{n} e^{\nu_i} \right]^{1/\|\beta\|} = e^{\|\nu\|/\|\beta\|} \to 1 \quad .$$

Therefore,

$$\overline{\lim}_{\beta\to\infty} \frac{|c_{\beta+\nu}|^{m/\|\beta\|} A^{\beta/\|\beta\|} meL}{\beta^{\beta/\|\beta\|}} = \left(\frac{1}{s}\right)^1 \cdot \frac{1}{1} \cdot \frac{1}{1} \cdot 1 = \frac{1}{s} \quad .$$

We may now apply the result of (a) to deduce that each derived series (46) converges absolutely and uniformly on compact subsets of S_s. This conclusion legitimizes our formal termwise differentiations of the series, while also confirming the continuity of the sums.

Next, in view of (11), we compute the formal derivative of (39) with respect to t as

$$\partial_t \left(\sum_\beta \frac{c_\beta}{\beta!} \, p_\beta(x,t) \right) = \sum_\beta \sum_{\alpha\leq\beta} c_\beta a_\alpha(t) \, \frac{p_{\beta-\alpha}(x,t)}{(\beta-\alpha)!} \quad , \tag{47}$$

where

$$\sum_\beta \sum_{\alpha\leq\beta} \left| c_\beta a_\alpha(t) \, \frac{p_{\beta-\alpha}(x,t)}{(\beta-\alpha)!} \right| = \sum_\alpha |a_\alpha(t)| \sum_\beta |c_{\beta+\alpha}| \, \frac{|p_\beta\,(x,t)|}{\beta!} \quad .$$

As above, each of the series

$$\sum_\beta |c_{\beta+\alpha}| \, \frac{|p_\beta(x,t)|}{\beta!}$$

converges uniformly and absolutely on compact subsets of S_s; as the coefficients $\{a_\alpha(t)\}$ are bounded and finite in number, we have the same convergence for the series (47). Thus termwise differentiation is justified in (47), and the sum of the series is continuous in S_s.

Applying again (10) and (11), we compute also formal mixed derivatives of (39),

$$\partial_t \partial_x^\nu \sum_\beta \frac{c_\beta}{\beta!} p_\beta(x,t) = \partial_x^\nu \partial_t \sum_\beta \frac{c_\beta}{\beta!} p_\beta(x,t) = \sum_\beta \sum_{\alpha \le \beta} c_{\beta+\nu} a_\alpha(t) \frac{p_{\beta-\alpha}(x,t)}{(\beta-\alpha)!} \quad ,$$

where

$$\sum_\beta \sum_{\alpha \le \beta} \left| c_{\beta+\nu} a_\alpha(t) \frac{p_{\beta-\alpha}(x,t)}{(\beta-\alpha)!} \right| = \sum_\alpha |a_\alpha(t)| \sum_\beta |c_{\beta+\alpha+\nu}| \frac{|p_\beta(x,t)|}{\beta!} \quad .$$

By arguments above, these series likewise converge absolutely and uniformly on compact subsets of S_s; thus termwise differentiation is justified and continuity of the sums confirmed.

The formula $\mathcal{L}u = 0$ follows by termwise differentiation and the fact that $\mathcal{L}p_\beta = 0$ for each β. Finally, setting $(x,t) = (0,0)$ in (46) and using $p_\beta(x,0) = x^\beta$ gives $\partial_x^\nu u(0,0) = c_\nu$, and thereby (41). $\diamondsuit$

It is noteworthy in formula (40) that bounds on the lower order coefficients $\{a_\alpha\}$ do not appear; that is, only bounds $\{A_\alpha\}$ with $|\alpha| = m$ affect the estimate on the width of the strip of convergence.

In the case of one space dimension, when $n = 1$, there are no mixed derivatives in the operator (1); then A is a scalar, β is an integer, and (40) reduces to

$$\overline{\lim}_{\beta\to\infty} \frac{|c_\beta|^{m/\beta} meA}{\beta} = \frac{1}{s} \quad . \tag{48}$$

For the special case of the Kemnitz operator,

$$\partial_t u(x,t) - \partial_x^m u(x,t) \quad ,$$

we have $A = 1$ and (48) becomes the estimate of Kemnitz [19].

For a special class of operators we can demonstrate that formula (40) gives the precise width of the strip of convergence.

Theorem 7 *Under the hypotheses of Theorem 5, suppose further that $\mathcal{L}$ has the form*

$$\mathcal{L}u(x,t) = \partial_t u(x,t) - \sum_{i=1}^n a_i \frac{\partial^m u(x,t)}{\partial {x_i}^m} - \sum_{\|\alpha\|<m} a_\alpha(t) \partial_x^\alpha u(x,t) \quad ,$$

where there are no mixed derivatives of highest order m, and coefficients of the unmixed derivatives of order m are constant. Assume also that either

(a) all coefficients of space derivatives are nonnegative, or

(b) all coefficients of space derivatives are nonpositive.

Then the formula of Theorem 6,

$$\overline{\lim}_{\beta\to\infty} \frac{|c_\beta|^{m/\|\beta\|}\, meA^{\beta/\|\beta\|}}{\beta^{\beta/\|\beta\|}} = \frac{1}{s} \quad , \tag{49}$$

where $A = (|a_1|, |a_2|, \cdots, |a_n|)$*, gives the largest possible width of the strip of convergence* S_s *of the series*

$$\sum_\beta \frac{c_\beta}{\beta!}\, p_\beta(x,t) \quad . \tag{50}$$

More precisely, if $0 < s < s'$ *and if* I *contains the interval* $(-s', s')$*, then there is a point* (x,t) *in the strip* $S_{s'}$ *where (50) does not converge.*

Proof: We assume q is positive and small enough that $0 < s < s+q < s'$, where $(-s', s') \subset I$. Choosing $x = (1, \cdots, 1)$ in (9) gives

$$\frac{p_\beta(x,t)}{\beta!} = \sum_{\overline{\alpha}\cdot\sigma\le\beta} \frac{b(t)^\sigma}{(\beta - \overline{\alpha}\cdot\sigma)!\sigma!} \quad . \tag{51}$$

For each α, definition (7) shows that $b_\alpha(t) \ge 0$ for $t \ge 0$ under condition (a), and $b_\alpha(t) \ge 0$ for $t \le 0$ under condition (b). Accordingly we choose $t = s+q$ under condition (a), and $t = -s-q$ under condition (b). Then $(x,t) \in S_{s'}$, each term in (51) is nonnegative, and the sum is at least as big as any single term. We consider the term corresponding to the multi-index $\sigma = \{\sigma_\alpha\}$, where

$$\sigma_\alpha = \begin{cases} [\beta_i/m] \quad , & \text{if } \alpha = me_i \quad , \\ 0 \quad , & \text{otherwise} \quad . \end{cases}$$

(The notation $[y]$ refers to the greatest integer no larger then y.) Then for this term,

$$\overline{\alpha}\cdot\sigma = \sum_{i=1}^n m\,[\beta_i/m]\, e_i \le \sum_{i=1}^n \beta_i e_i = \beta \quad ,$$

and

$$b(t)^\sigma = \prod_{i=1}^n b_i(t)^{[\beta_i/m]} \quad .$$

Because each a_i is constant, (7) and (17) show that $b_i(t) = a_i t$ and $A_i = |a_i|$. Letting $[\beta/m]$ denote the vector

$$[\beta/m] = ([\beta_1/m], \cdots, [\beta_n/m]) \quad ,$$

we may write

$$0 \le b(t)^\sigma = \prod_{i=1}^n a_i^{[\beta_i/m]} t^{[\beta_i/m]} = A^{[\beta/m]}\,(s+q)^{\|[\beta/m]\|} \quad ,$$

$$\sigma! = [\beta/m]! \quad , \quad \overline{\alpha}\cdot\sigma = m\,[\beta/m] \quad .$$

As (51) is bounded below by the corresponding single term, we have for these chosen values of x and t the lower bound

$$\frac{p_\beta\,(x,t)}{\beta!} \ge \frac{A^{[\beta/m]}\,(s+q)^{\|[\beta/m]\|}}{(\beta - m\,[\beta/m])!\,[\beta/m]!} \quad . \tag{52}$$

We investigate the individual factors on the right side of (52). First, given any nonnegative integer k, we may write

$$k = \ell m + r \quad , \tag{53}$$

where $\ell = [k/m]$ and $0 \le r < m$. Then, given $\rho > 0$,

$$\rho^{[k/m]} = \rho^\ell = \frac{\rho^{k/m}}{\rho^{r/m}} \ge \frac{\rho^{k/m}}{1+\rho} \quad .$$

Consequently, in (52) we have

$$A^{[\beta/m]} = \prod_{i=1}^n A_i^{[\beta_i/m]} \ge \prod_{i=1}^n \frac{A_i^{\beta_i/m}}{1+A_i} \ge \frac{A^{\beta/m}}{(1+\|A\|)^n} \quad , \tag{54}$$

and

$$(s+q)^{\|[\beta/m]\|} = \prod_{i=1}^n (s+q)^{[\beta_i/m]} \ge \prod_{i=1}^n \frac{(s+q)^{\beta_i/m}}{(1+s+q)} = \frac{(s+q)^{\|\beta\|/m}}{(1+s+q)^n} \quad . \tag{55}$$

Returning back to (53), we have also

$$(k - m\,[k/m])! = (k-m\ell)! = r! \le m! \quad ,$$

and therefore

$$(\beta - m\,[\beta/m])! = \prod_{i=1}^n (\beta_i - m\,[\beta_i/m])! \le \prod_{i=1}^n m! = (m!)^n \quad . \tag{56}$$

Finally, if $k \ge m$ in (53) then $[k/m] \ge 1$ and $[k/m]! \le (k/m)!$, and use of the right side of (12) gives

$$[k/m]! \le e\sqrt{1+k/m}\left(\frac{k}{me}\right)^{k/m} \quad . \tag{57}$$

On the other hand, if $0 \le k < m$ in (53), then $[k/m]! = 0! = 1$, and (57) follows from the general bound $y^y \ge e^{-1/e}$ for $y \ge 0$, and the estimate

$$\left(\frac{k}{me}\right)^{k/m} = \left\{\left(\frac{k}{me}\right)^{k/(me)}\right\}^e \ge \left(e^{-1/e}\right)^e = 1/e \quad .$$

Several applications of (57) then yield

$$\begin{aligned}[\beta/m]! &= \prod_{i=1}^{n}[\beta_i/m]! \le \prod_{i=1}^{n} e\sqrt{1+\beta_i/m}\left(\tfrac{\beta_i}{me}\right)^{\beta_i/m} \\ &\le e^n\left(1+\|\beta\|/m\right)^{n/2}\frac{\beta^{\beta/m}}{(me)^{\|\beta\|/m}} \quad .\end{aligned} \tag{58}$$

Substituting now (54), (55), (56), and (58) into (52), we arrive at

$$\frac{p_\beta(x,t)}{\beta!} \ge \frac{C(m,n,A,s+q)}{\left(1+\|\beta\|/m\right)^{n/2}}\frac{A^{\beta/m}}{\beta^{\beta/m}}\left\{me(s+q)\right\}^{\|\beta\|/m} \quad .$$

Next we observe from the hypotheses (49) that there is an infinite collection of multi-indices $\{\beta\}$ such that

$$\frac{|c_\beta|^{m/\|\beta\|}\, meA^{\beta/\|\beta\|}}{\beta^{\beta/\|\beta\|}} \ge \frac{1}{s+q/2} \quad ,$$

or equivalently,

$$|c_\beta| \ge \frac{\beta^{\beta/m}}{A^{\beta/m}}\left[\frac{1}{me(s+q/2)}\right]^{\|\beta\|/m} \quad .$$

Therefore, for multi-indices β in this infinite collection,

$$\left|c_\beta\frac{p_\beta(x,t)}{\beta!}\right| \ge \frac{C(m,n,A,s+q)}{\left(1+\|\beta\|/m\right)^{n/2}}\left[\frac{s+q}{s+q/2}\right]^{\|\beta\|/m} \quad .$$

Since the last expression on the right is unbounded as $\|\beta\| \to \infty$, the series (50) cannot converge at our point $(x,t) = (1,\cdots,1,t)$, where $t = s+q$ in case (a) and $t = -s-q$ case (b). This point lies in $S_{s'}$. $\diamondsuit$

We point out that our formula (49) for the width of the strip of convergence specializes to the formula of Rosenbloom and Widder [21, 22, 24] for the heat equation, and to that of Kemnitz [19] for his generalization of the one-dimensional heat equation. The result of Rosenbloom and Widder is stated for a modified version of the series (50),

$$\sum_\beta a_\beta p_\beta(x,t) \quad .$$

Our formula (49) when specialized to the heat equation, and the Rosenbloom – Widder formula are, respectively,

$$\overline{\lim}_{\beta\to\infty}\frac{|c_\beta|^{2/\|\beta\|}\,2e}{\beta^{\beta/\|\beta\|}} = \frac{1}{s} \quad , \quad \overline{\lim}_{\beta\to\infty}\frac{|a_\beta|^{2/\|\beta\|}\,2\beta^{\beta/\|\beta\|}}{e} = \frac{1}{s} \quad .$$

With the identification $a_\beta = c_\beta/\beta!$, and with the help of (12), it is not difficult to demonstrate that these formulas are equivalent.

6. Cauchy Problems

Again we let $\mathcal{L}$ be the operator (1). We consider the Cauchy problem

$$\begin{cases} \mathcal{L}u(x,t) = 0 & , \\ u(x,0) = f(x) & . \end{cases} \tag{59}$$

We assume f is a real valued function on R^n, with a power series expansion in R^n which we write in the two ways

$$f(x) = \sum_\beta a_\beta x^\beta = \sum_\beta \frac{c_\beta}{\beta!}\, x^\beta \quad . \tag{60}$$

We introduce some terminology regarding growth conditions on the function f. We state the conditions for complex valued analytic functions defined on C^n, as they are more natural in that setting. Whereas for entire functions of a single complex variable there is general agreement on definitions of order and type (see [2], for example), for functions of several complex variables there is no standard terminology, as there are several possible definitions depending on one's choice of norms in C^n and other considerations. (See [6], Chapter 1, for a discussion of this topic.) Although it appears that all definitions give the same order of an entire function, the type varies with the definition. The definition below is most suitable for our purposes here, as it allows the natural extension to higher space dimensions of one-dimensional results of Rosenbloom and Widder, and later of Kemnitz, regarding the Cauchy problem (59). We explain in Theorem 9 how the prescribed conditions on the coefficients of the power series expansion of an entire function are equivalent to growth limits at infinity on the function.

Definition 8 *Let $f = f(z) = f(z_1, z_2, \cdots, z_n)$ be an entire function f in C^n, with the power series expansion (with complex coefficients)*

$$f(z) = \sum_\beta a_\beta z^\beta = \sum_\beta \frac{c_\beta}{\beta!}\, z^\beta \quad . \tag{61}$$

Given ρ with $0 < \rho < \infty$ and a vector $B = (B_1, B_2, \cdots, B_n)$ of positive real numbers, we say that f has growth $[\rho, B]$ provided that

$$\overline{\lim}_{\beta\to\infty} \frac{\beta^{\beta/\|\beta\|}\, |a_\beta|^{\rho/\|\beta\|}}{B^{\beta/\|\beta\|}} \le e\rho \quad . \tag{62}$$

It is readily seen that if $\rho \le \tau$ and $B \le C$, and if f has growth $[\rho, B]$, then f has growth $[\tau, C]$. (One verifies first that (62) implies $|a_\beta| \to 0$ as $\|\beta\| \to \infty$.) This monotonicity property follows also directly from Theorem 9 below (with the observation $|z_i|^\rho \le 1 + |z_i|^\tau$).

Theorem 9 *Assume $0 < \rho < \infty$, that B is an n-vector of positive real numbers, and $a_\beta = c_\beta/\beta!$ for each β. Then condition (62) is equivalent to*

$$\overline{\lim}_{\beta\to\infty} \left(\frac{e}{\beta^{\beta/\|\beta\|}}\right)^{\rho-1} \frac{|c_\beta|^{\rho/\|\beta\|}}{B^{\beta/\|\beta\|}} \leq \rho \ , \tag{63}$$

and under these conditions on the coefficients the series (61) converges to an entire function f in C^n satisfying the growth condition

$$|f(z)| = \mathcal{O}\left(\exp\left[\eta \sum_{i=1}^{n} B_i |z_i|^\rho\right]\right) \quad \text{as } z \to \infty, \text{ for all } \eta > 1 \ . \tag{64}$$

Conversely, if f is an entire function in C^n with growth condition (64) and the expansion (61), then (62) and (63) must hold.

Before proving Theorem 9 we demonstrate how the coefficient conditions of that theorem pertain to the Cauchy problem (59). A real valued function f with an expansion (60) in R^n can be viewed as the restriction to R^n of an analytic function in C^n with the expansion (61). Accordingly, we may apply Definition 8 just as well to real valued functions with expansions (60) valid in R^n.

Theorem 10 *Under the hypotheses of Theorem 5 assume that the function (60) has growth $[m/(m-1), B]$, where*

$$B = \tau\left(\frac{1}{{A_1}^{1/(m-1)}}, \frac{1}{{A_2}^{1/(m-1)}}, \cdots, \frac{1}{{A_n}^{1/(m-1)}}\right) \tag{65}$$

and $0 < \tau < \infty$. Then the series

$$u(x,t) = \sum_\beta \frac{c_\beta}{\beta!} p_\beta(x,t) \tag{66}$$

solves the Cauchy problem (59) in the strip

$$S_s = \{(x,t) : x \in R^n \ , \ t \in I \ , \ |t| < s\} \ ,$$

where

$$s = \frac{1}{m}\left(1 + \sum_{\alpha\in\mathcal{A} \ , \ \alpha \text{ mixed} \ , \ \|\alpha\|=m} \frac{A_\alpha}{\varepsilon^{\alpha/m}}\right)^{-1} \left(\frac{m-1}{m\tau}\right)^{m-1} \ . \tag{67}$$

Moreover, the series (66) and its sum $u(x,t)$ have the convergence and regularity properties described in Theorem 6.

Proof: Under the hypotheses of Theorem 5, and the assumption of validity of Theorem 9, the equivalent growth conditions (62) and (63) on f, with $\rho = (m-1)/m$ and B prescribed by (65), imply

$$\overline{\lim}_{\beta\to\infty} \frac{|c_\beta|^{m/\|\beta\|}\, meA^{\beta/\|\beta\|}}{\beta^{\beta/\|\beta\|}} \left(1 + \sum_{\alpha\in\mathcal{A}\ ,\ \alpha \text{ mixed}\ ,\ \|\alpha\|=m} \frac{A_\alpha}{\varepsilon^{\alpha/m}}\right) \le \frac{1}{s} \ ,$$

where s is given by (67). Thus Theorem 6 applies, confirming the stated convergence properties of the series (66) and the regularity properties of its sum $u(x,t)$, and affirming as well the equation $\mathcal{L}u = 0$ in S_s. The initial value of u is

$$u(x,0) = \sum_\beta \frac{c_\beta}{\beta!}\, p_\beta(x,0) = \sum_\beta \frac{c_\beta}{\beta!}\, x^\beta = f(x) \ . \qquad \diamondsuit$$

In [17] the following technical lemma is proved.

Lemma 11 *If* $0 < r, \rho < \infty$, *then*

$$\sum_{k=0}^{\infty} \frac{r^k}{k^{k/\rho}} \le (2^\rho r^\rho + 2)\exp\left(\frac{r^\rho}{e\rho}\right) \ .$$

Proof of Theorem 9: We use techniques adapted from [2, 6, 22].

First, from (12) it follows that for any multi-index β in R^n,

$$\beta^\beta e^{-\|\beta\|} \le \beta! \le \beta^\beta e^{-\|\beta\|} e^n \prod_{i=1}^n \sqrt{\beta_i + 1} \ .$$

From this inequality and the observation

$$\lim_{\|\beta\|\to\infty} \left(e^n \prod_{i=1}^n \sqrt{\beta_i+1}\right)^{\rho/\|\beta\|} = 1 \ ,$$

it is readily checked that (62) is equivalent to (63).

Next we assume (62) holds on the coefficients of (61), and we show that the series converges to an entire function f satisfying (64). Given fixed $\eta > 1$, (62) implies that for some positive constant M and for all β,

$$|a_\beta| \le M \frac{(\eta e\rho)^{\|\beta\|/\rho} B^{\beta/\rho}}{\beta^{\beta/\rho}} \ .$$

Then for the series (61) we have the bound

$$|f(z)| \ \le \ M \sum_\beta \frac{(\eta e\rho)^{\|\beta\|/\rho} B^{\beta/\rho}}{\beta^{\beta/\rho}} \left|z^\beta\right|$$

$$
\begin{aligned}
&= M \sum_{\beta_1,\beta_2,\cdots,\beta_n=0}^{\infty} \prod_{i=1}^{n} \frac{\left\{(\eta e\rho B_i)^{1/\rho} |z_i|\right\}^{\beta_i}}{\beta_i^{\beta_i/\rho}} \\
&= M \prod_{i=1}^{n} \sum_{\beta_i=0}^{\infty} \frac{\left\{(\eta e\rho B_i)^{1/\rho} |z_i|\right\}^{\beta_i}}{\beta_i^{\beta_i/\rho}} \quad .
\end{aligned}
$$

We apply Lemma 11 to each of the n sums to obtain

$$
\begin{aligned}
|f(z)| &\leq M \prod_{i=1}^{n} (2^\rho \eta e\rho B_i |z_i|^\rho + 2) \exp[\eta B_i |z_i|^\rho] \\
&= M \exp\left[\eta \sum_{i=1}^{n} B_i |z_i|^\rho\right] \prod_{i=1}^{n} (2^\rho \eta e\rho B_i |z_i|^\rho + 2) \\
&= \mathcal{O}\left(\exp\left[\eta^2 \sum_{i=1}^{n} B_i |z_i|^\rho\right]\right) \quad .
\end{aligned}
$$

Replacing η with $\sqrt{\eta}$, we obtain (64). Clearly f, the sum in C^n of a power series, is entire.

Finally, we suppose that f is entire in C^n with the expansion (61), and that (64) holds, and we verify (62). Given $\eta > 1$, (64) implies there is a positive constant M such that

$$
|f(z)| \leq M \exp\left[\eta \sum_{i=1}^{n} B_i |z_i|^\rho\right] \quad . \tag{68}
$$

By the generalized Cauchy inequality for analytic functions in C^n (see [18], Theorem 2.2.7), for any multi-index β and positive vector $r = (r_1, \cdots, r_n)$,

$$
|c_\beta| = \left|\partial^\beta f(0)\right| \leq \frac{\beta!}{r^\beta} \sup\{|f(z)| : |z_i| \leq r_i \ , \ 1 \leq i \leq n\} \quad .
$$

With the bound (68), this inequality leads to

$$
|c_\beta| \leq M\beta! \prod_{i=1}^{n} {r_i}^{-\beta_i} \exp[\eta B_i {r_i}^\rho] \quad . \tag{69}
$$

We choose the positive constants r_i so that the right side of this inequality is minimized. If $\beta_i > 0$ we choose $r_i = \{\beta_i / (\rho\eta B_i)\}^{1/\rho}$ to obtain

$$
{r_i}^{-\beta_i} \exp[\eta B_i {r_i}^\rho] = \frac{(e\rho\eta B_i)^{\beta_i/\rho}}{\beta_i^{\beta_i/\rho}} \quad ,
$$

while if $\beta_i = 0$ we observe that

$$
\lim_{r_i \to 0^+} {r_i}^{-\beta_i} \exp[\eta B_i {r_i}^\rho] = 1 = \frac{(e\rho\eta B_i)^{\beta_i/\rho}}{\beta_i^{\beta_i/\rho}} \quad .
$$

Thus, (69) implies

$$|a_\beta| = \frac{|c_\beta|}{\beta!} \leq M \prod_{i=1}^{n} \frac{(e\rho\eta B_i)^{\beta_i/\rho}}{\beta_i^{\beta_i/\rho}} = M\,(e\rho\eta)^{\|\beta\|/\rho}\,\frac{B^{\beta/\rho}}{\beta^{\beta/\rho}} \quad .$$

From this inequality it follows that

$$\overline{\lim}_{\beta\to\infty} \frac{\beta^{\beta/\|\beta\|}\,|a_\beta|^{\rho/\|\beta\|}}{B^{\beta/\|\beta\|}} \leq \eta e\rho \ \ .$$

We let η approach 1 from above to obtain (62). $\diamondsuit$

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