GRADIENT BOUNDS FOR HARMONIC FUNCTIONS LIPSCHITZ ON THE BOUNDARY

G. N. HILE AND ALEXANDER STANOYEVITCH

Dedicated to our friend and colleague Heinrich Begehr

ABSTRACT. We estimate the magnitude of the gradient of the solution to the Dirichlet problem, with Lipschitz boundary data, for the Laplace equation on a bounded domain Ω in $\mathbb{R}^n$. Under certain regularity assumptions on $\partial\Omega$, for $x \in \Omega$ we establish the estimate

$$|\nabla u(x)| \leq C(n,\Omega) M \log \frac{diameter(\Omega)}{dist(x,\partial\Omega)} \ ,$$

where M is the Lipschitz constant of the boundary data.

1. INTRODUCTION

Let Ω be a domain in $\mathbb{R}^n$, $n \geq 2$, and let $\varphi : \partial\Omega \longrightarrow \mathbb{R}$ be bounded and Lipschitz continuous on $\partial\Omega$. G. Hardy and J. Littlewood [4], [5] established the following estimate when Ω is the upper half space

$$\mathbb{R}^n_+ := \{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_n > 0\} \quad .$$

Theorem 1 (Hardy & Littlewood)**.** *If $\Omega = \mathbb{R}^n_+$ and φ is bounded and Lipschitz continuous on $\partial\Omega$, then the bounded solution $u \in C\left(\overline{\Omega}\right)$ of the Dirichlet problem $\Delta u = 0$ in Ω, $u = \varphi$ on $\partial\Omega$ satisfies*

$$|\nabla u(x)| = O\left(\log 1/x_n\right) \quad , \quad as\ x_n \to 0^+ \ . \tag{1.1}$$

We will obtain a more precise version of (1.1) for bounded domains Ω, with x_n replaced by the distance from x to $\partial\Omega$.

Recall that a domain Ω is a C^k*-domain*, where k is a nonnegative integer, if $\partial\Omega$ is locally representable by graphs of C^k functions in $n-1$ variables. As is well known, for bounded C^1 domains the Dirichlet problem $u \in C(\overline{\Omega})$, $\Delta u = 0$ in Ω, $u = \varphi$ on $\partial\Omega$ is uniquely solvable provided φ is continuous on $\partial\Omega$.

The following theorem is our main result. (We let $d(\Omega)$ denote the diameter of Ω, and $d(x,\partial\Omega)$ the distance from x to $\partial\Omega$.)

Theorem 2. *Let Ω be a bounded C^2-domain in $\mathbb{R}^n$, and let $\varphi : \partial\Omega \longrightarrow \mathbb{R}$ satisfy the Lipschitz condition*

$$|\varphi(z) - \varphi(\zeta)| \leq M\,|z - \zeta| \quad for\ all\ z, \zeta \in \partial\Omega \ .$$

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Let u solve the Dirichlet problem $u \in C\left(\overline{\Omega}\right)$, $\Delta u = 0$ in Ω, $u = \varphi$ on $\partial\Omega$. Then there exists a positive constant $C(n, \Omega)$ such that, for all x in Ω,

$$|\nabla u(x)| \leq C(n, \Omega) M \log \frac{d(\Omega)}{d(x, \partial\Omega)} \quad . \tag{1.2}$$

In the last section of the paper we show how, in dimension $n = 2$ and in the case of Jordan domains, the C^2 condition can be relaxed to a Dini-smooth condition on $\partial\Omega$.

Perhaps the first significant results concerning the behaviour of the derivatives of a harmonic function near the boundary of a domain were established by Kellogg [7]. He showed that, (summarizing roughly) in a three dimensional domain near a boundary element $z = f(x, y)$, the first order derivatives of a harmonic function u are continuous up to the boundary provided that the first order derivatives of f and of u on the boundary itself are Dini continuous; moreover, if the first order derivatives of f and u on the boundary are Hölder continuous with exponent less than 1, then ∇u is Hölder continuous up to the boundary. The well-known Schauder theory (see chapter 6 of [3]) establishes analogous and more general results for uniformly elliptic partial differential equations, giving Hölder bounds for solutions u and their derivatives inside a domain in terms of Hölder norms of the boundary values of u and the nonhomogeneous term of the equation. These so-called "Schauder estimates" have been improved and extended by many authors, but generally for a second order equation boundary values are required to lie in the Hölder class $C^{2,\alpha}(\partial\Omega)$. D. Gilbarg and L. Hörmander [2] extended the Schauder theory to include conditions of lower regularity on the boundary values of the solution, as well as on the boundary of the domain and the coefficients of the equations. G. Troianiello [10] has weakened further some conditions of Gilbarg and Hörmander. However, it appears that none of these results, even for the special case of the Laplace equation, yield a logarithmic growth estimate on ∇u near the boundary when u is assumed only Lipschitz continuous on the boundary.

2. DOMAINS IN $\mathbb{R}^n$

We require a local and more quantitative version of the Hardy-Littlewood result of Theorem 1. For a point x in $\mathbb{R}^n_+$, we will sometimes decompose it as $x = (y, s)$ where $y = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$ and $s = x_n \in \mathbb{R}$. Given a bounded continuous function φ on $\mathbb{R}^{n-1}$, the bounded harmonic function u in $\mathbb{R}^n_+$, continuous in the closure of $\mathbb{R}^n_+$, and satisfying $u(y, 0) = \varphi(y)$ is uniquely determined and prescribed by the *Poisson integral formula*,

$$u(x) = \frac{2}{\omega_n} \int_{\mathbb{R}^{n-1}} \frac{s}{\left(|y - z|^2 + s^2\right)^{n/2}} \varphi(z)\, dz \quad , \tag{2.1}$$

where ω_n is the surface area of the unit ball in $\mathbb{R}^n$. (See for example [1] or [9].)

Lemma 1. *Suppose φ is measurable and real valued on $\mathbb{R}^{n-1}$, and that there are constants $K \geq 0$, $M \geq 0$, $\epsilon > 0$ such that, for $z \in \mathbb{R}^{n-1}$,*
(i) $|\varphi(z)| \leq K$,
(ii) $|\varphi(z) - \varphi(0)| \leq M\,|z|$ *for* $|z| < \epsilon$.
Then there are positive constants $C(n)$ and $C(n, \epsilon)$ such that the Poisson integral u given by (2.1) satisfies

$$|\nabla u(0,s)| \leq C(n)\left[\frac{K}{\epsilon} + M \log\frac{1}{s}\right] \quad , \text{ for } 0 < s < C(n,\epsilon) \quad . \tag{2.2}$$

Proof. First, for $s > 0$ we use (2.1) to compute

$$\begin{aligned}\frac{\partial u(0,s)}{\partial s} &= \frac{2}{\omega_n}\int_{\mathbb{R}^{n-1}} \left(|z|^2 + (1-n)s^2\right)\left(|z|^2 + s^2\right)^{-1-n/2} \varphi(z)\, dz \\ &= I(s) + J(s) + L(s) \quad ,\end{aligned}$$

where

$$\begin{aligned} I(s) &= \frac{2}{\omega_n}\int_{|z|\geq\epsilon} \left(|z|^2 + (1-n)s^2\right)\left(|z|^2 + s^2\right)^{-1-n/2} \varphi(z)\, dz \quad , \\ J(s) &= \frac{2}{\omega_n}\int_{|z|\leq\epsilon} \left(|z|^2 + (1-n)s^2\right)\left(|z|^2 + s^2\right)^{-1-n/2} [\varphi(z) - \varphi(0)]\, dz \quad , \\ L(s) &= \frac{2}{\omega_n}\int_{|z|\leq\epsilon} \left(|z|^2 + (1-n)s^2\right)\left(|z|^2 + s^2\right)^{-1-n/2} \varphi(0)\, dz \quad . \end{aligned}$$

To estimate these integrals we will use the identity

$$\left(|z|^2 + (1-n)s^2\right)\left(|z|^2 + s^2\right)^{-1-n/2} = -\nabla_z \cdot \left[z\left(|z|^2 + s^2\right)^{-n/2}\right] \quad .$$

We assume $0 < s \leq \epsilon/\sqrt{n-1}$, so that $|z|^2 + (1-n)s^2 \geq 0$ in $I(s)$. Using assumption (i), and letting $d\sigma$ signify integration with respect to surface measure, we find that

$$\begin{aligned} |I(s)| &\leq C(n)K \int_{|z|\geq\epsilon} \left(|z|^2 + (1-n)s^2\right)\left(|z|^2 + s^2\right)^{-1-n/2} dz \\ &= C(n)K \lim_{R\to\infty}\int_{\epsilon\leq|z|\leq R} -\nabla_z \cdot \left[z\left(|z|^2 + s^2\right)^{-n/2}\right] dz \\ &= C(n)K \lim_{R\to\infty}\left[\int_{|z|=\epsilon} - \int_{|z|=R}\right]\left[z\left(|z|^2 + s^2\right)^{-n/2}\right] \cdot \frac{z}{|z|}\, d\sigma(z) \\ &= C(n)K \int_{|z|=\epsilon} \epsilon\left(\epsilon^2 + s^2\right)^{-n/2} d\sigma(z) \leq C(n)K/\epsilon \quad . \end{aligned}$$

(When $n = 2$ the surface integrals above are replaced by endpoint evaluations, but the end result is the same.)

With similar calculations, we produce the bound

$$\begin{aligned} |L(s)| &\leq C(n)K \left| \int_{|z|\leq\epsilon} \nabla_z \cdot \left[z \left(|z|^2 + s^2 \right)^{-n/2} \right] \, dz \right| \\ &= C(n)K \left| \int_{|z|=\epsilon} \left[z \left(|z|^2 + s^2 \right)^{-n/2} \right] \cdot \frac{z}{|z|} \, d\sigma(z) \right| \\ &= C(n)K \int_{|z|=\epsilon} \epsilon \left(\epsilon^2 + s^2 \right)^{-n/2} \, d\sigma(z) \leq C(n)K/\epsilon \ . \end{aligned}$$

Finally, still assuming $0 < s \leq \epsilon/\sqrt{n-1}$, we use (ii) to estimate

$$\begin{aligned} |J(s)| &\leq C(n)M \int_{|z|\leq\epsilon} \left| |z|^2 + (1-n)s^2 \right| \left(|z|^2 + s^2 \right)^{-1-n/2} |z| \, dz \\ &\leq C(n)M \int_{|z|\leq\epsilon} |z| \left(|z|^2 + s^2 \right)^{-n/2} \, dz \\ &\leq C(n)M \int_0^\epsilon \rho^{n-1} \left(\rho^2 + s^2 \right)^{-n/2} \, d\rho \\ &\leq C(n)M \int_0^\epsilon \rho^{n-1} \left(\rho^n + s^n \right)^{-1} \, d\rho \\ &\leq C(n)M \left[\log\left(\epsilon^n + s^n \right) - \log s^n \right] \\ &\leq C(n)M \left[\log\left(2\epsilon^n \right) + n \log \frac{1}{s} \right] \ . \end{aligned}$$

Combining our three estimates we conclude that, for $0 < s \leq \epsilon/\sqrt{n-1}$,

$$\left| \frac{\partial u(0,s)}{\partial s} \right| \leq C(n)\frac{K}{\epsilon} + C(n)M \left[\log\left(2\epsilon^n \right) + n \log \frac{1}{s} \right] \ .$$

Next, from (2.1) we calculate

$$\nabla_y u(0,s) = ns\frac{2}{\omega_n} \int_{\mathbb{R}^{n-1}} z \left(|z|^2 + s^2 \right)^{-1-n/2} \varphi(z) \, dz \ .$$

With the observation

$$\int_{|z|\leq\epsilon} z \left(|z|^2 + s^2 \right)^{-1-n/2} \, dz = 0 \ ,$$

we may write

$$\begin{aligned} \nabla_y u(0,s) = {} & ns\frac{2}{\omega_n} \int_{|z|>\epsilon} z \left(|z|^2 + s^2 \right)^{-1-n/2} \varphi(z) \, dz \\ & + ns\frac{2}{\omega_n} \int_{|z|\leq\epsilon} z \left(|z|^2 + s^2 \right)^{-1-n/2} \left[\varphi(z) - \varphi(0) \right] \, dz \ , \end{aligned}$$

and use (i) and (ii) to estimate

$$\begin{aligned}
|\nabla_y u(0,s)| &\le C(n)Ks\int_{|z|>\epsilon} |z|^{-1-n}\ dz \\
&\quad + C(n)Ms\int_{|z|\le\epsilon} |z|^2\left(|z|^2+s^2\right)^{-1-n/2}\ dz \\
&\le C(n)\frac{Ks}{\epsilon^2} + C(n)M\int_{\mathbb{R}^{n-1}} |y|^2\left(|y|^2+1\right)^{-1-n/2}\ dy \\
&\le C(n)\frac{K\epsilon/\sqrt{n-1}}{\epsilon^2} + C(n)M \le C(n)\frac{K}{\epsilon} + C(n)M \quad .
\end{aligned}$$

In summary, writing $\nabla u = (\nabla_y u, \partial_s u)$, we have verified that, for $0 < s \le \epsilon/\sqrt{n-1}$,

$$|\nabla u(0,s)| \le C(n)\frac{K}{\epsilon} + C(n)M\left[n\log\frac{1}{s} + \log\left(2\epsilon^n\right) + 1\right] \quad .$$

Our desired estimate (2.2) follows, provided that $0 < s \le C(n,\epsilon)$ for some suitably small constant $C(n,\epsilon)$. □

Our next step is to transfer the above result to balls. We let e_n denote the unit vector $(0,\dots,0,1)$ in $\mathbb{R}^n$.

Lemma 2. *Suppose φ is continuous and real valued on the boundary of an open ball B of radius a in $\mathbb{R}^n$, let p be a point on ∂B, and suppose further that there are constants K, M, and ϵ $(0 < \epsilon \le 2a)$ such that, for z on ∂B,*

(i) $|\varphi(z)| \le K$,

(ii) $|\varphi(z) - \varphi(p)| \le M\,|z-p|$ for $|z-p| < \epsilon$.

Let u be the solution of the Dirichlet problem $u \in C\left(\overline{B}\right)$, $\Delta u = 0$ in B, $u = \varphi$ on ∂B. Then there are positive constants $C(n)$ and $C(n,\epsilon,a)$ such that, for x on the radial line between p and the center of B and $0 < d(x,\partial B) < C(n,\epsilon,a)$,

$$|\nabla u\,(x)| \le C(n)\left[\frac{K}{\epsilon} + M\log\frac{2a}{d(x,\partial B)}\right] \quad . \tag{2.3}$$

Proof. First we consider a ball B of diameter 1. For convenience we translate and rotate B so that its center is $e_n/2$ and the boundary point of interest is the top point $p = e_n$. (Such a change of variables does not alter the Laplacian of u nor the magnitude of its gradient.) Without loss of generality we may assume $\varphi(p) = 0$, since subtracting the constant $\varphi(p)$ from φ (and hence also from u) will not alter the constant M in (ii) and will at most double the constant K of (i), while the gradient of u will remain the same.

Let ρ be reflection with respect to the unit sphere centered at the origin in $\mathbb{R}^n$; i.e, $\rho(x) = x/\left|x\right|^2$ for $x \in \mathbb{R}^n\backslash\{0\}$. It is not hard to verify that $\rho(B) = \mathcal{U}$, where $\mathcal{U}$ is the half space consisting of points in $\mathbb{R}^n$ whose last coordinate is larger than 1. Also, $\rho\left(\partial B - \{0\}\right) = \partial\mathcal{U}$ and $\rho(p) = p$. Note also that $\rho = \rho^{-1}$ so that $\rho(\mathcal{U}) = B$. Since u is harmonic in B, its *Kelvin transform*,

$$v(x) := |\rho(x)|^{n-2} u(\rho(x)) = |x|^{2-n} u\left(x/|x|^2\right) \quad ,$$

is harmonic in $\mathcal{U}$. We define ψ on $\partial\mathcal{U}$ by

$$\psi(z) := |\rho(z)|^{n-2} \varphi(\rho(z)) = |z|^{2-n} \varphi\left(z/|z|^2\right) \quad ;$$

then v is a solution in $\mathcal{U}$ of the Dirichlet problem $v \in C\left(\overline{\mathcal{U}}\right)$, $\Delta v = 0$ in $\mathcal{U}$, $v = \psi$ on $\partial\mathcal{U}$. Moreover, since $|x| \geq 1$ on $\overline{\mathcal{U}}$ we have both $|\psi| \leq K$ on $\partial\mathcal{U}$ and $|v| \leq K$ on $\overline{\mathcal{U}}$.

In order to apply Lemma 1 to v, we need an analog of (ii) to hold for ψ. If $z \in \overline{\mathcal{U}}$ and $|z - p| < \epsilon/3$ then, using $|z| \geq 1$ and $|p| = 1$, we find that

$$\begin{aligned}
|\rho(z) - \rho(p)| &= \left| z|z|^{-2} - p|p|^{-2} \right| \\
&= |z|^{-2}|p|^{-2}\left| z\left(|p|^2 - |z|^2\right) + (z-p)|z|^2 \right| \\
&\leq |z|^{-2}|p|^{-2}\left[|z|\,|p-z|\,(|p| + |z|) + |z-p|\,|z|^2 \right] \\
&= \left[|z|^{-1}|p|^{-1} + 2|p|^{-2} \right] |z-p| \leq 3|z-p| < \epsilon \quad ;
\end{aligned}$$

therefore, using (ii) and $\varphi(p) = \psi(p) = 0$, for $|z - p| < \epsilon/3$ and $z \in \partial\mathcal{U}$ we have

$$\begin{aligned}
|\psi(z) - \psi(p)| &= |\psi(z)| = |z|^{2-n}|\varphi(\rho(z))| \leq |\varphi(\rho(z))| \\
&= |\varphi(\rho(z)) - \varphi(\rho(p))| \leq M|\rho(z) - \rho(p)| \leq 3M|z - p| \quad .
\end{aligned}$$

If we translate $\mathcal{U}$ down one unit then $\mathcal{U}$ becomes the half space $x_n > 0$ and v becomes the Poisson integral of ψ. We apply Lemma 1 with ϵ replaced by $\epsilon/3$ and M by $3M$, and conclude that

$$\text{(2.4)} \quad |\nabla v(se_n)| \leq C(n)\left[\frac{K}{\epsilon} + M\log\frac{1}{s-1}\right] \quad , \text{ for } 0 < s - 1 < C(n,\epsilon) \quad .$$

A computation shows that

$$\text{(2.5)} \qquad \begin{aligned}
\nabla u(x) = (2-n)x|x|^{-n} v(\rho(x)) + |x|^{-n}\nabla v(\rho(x)) \\
- 2x|x|^{-2-n}\left[x \cdot \nabla v(\rho(x))\right] \quad .
\end{aligned}$$

We take $x = te_n$ with $1/2 < t < 1$; then $\rho(x) = e_n/t$, and we may use (2.4) and (2.5) to conclude that, if $0 < 1/t - 1 < C(n,\epsilon)$,

$$|\nabla u(te_n)| \leq (n-2)t^{1-n}K + 3t^{-n}C(n)\left[\frac{K}{\epsilon} + M\log\frac{1}{1/t - 1}\right] \quad .$$

As $1/2 < t < 1$ and $0 < \epsilon \leq 2a = 1$, this inequality leads to

$$|\nabla u(te_n)| \leq C(n)\left[\frac{K}{\epsilon} + M\log\frac{1}{1-t}\right] \quad .$$

Finally, since $d(x,\partial B) = |x - e_n| = 1-t$, and the condition $1/t-1 < C(n,\epsilon)$ reduces to $d(x,\partial B) < C(\eta,\epsilon)/\left(1 + C(n,\epsilon)\right)$, we have verified the lemma for balls of radius $1/2$.

To handle a ball B of general radius a we make a change of variables. We may assume B is centered at the origin. We define functions ψ and v as

$$\psi(z) = \varphi(2az) \quad , \quad v(x) = u(2ax) \ ,$$

so that ψ is defined on the boundary of the ball D of radius $1/2$ centered at the origin, and v solves the Dirichlet problem on this ball with boundary data ψ. Then $|\psi| \leq K$ and, setting $q = p/2a$, we have for $|z - q| < \epsilon/2a$ that $|2az - p| < \epsilon$, and hence

$$|\psi(z) - \psi(q)| = |\varphi(2az) - \varphi(p)| \leq M\,|2az - p| = 2aM\,|z - q| \ .$$

Applying the result for balls of radius $1/2$ to v, with ϵ replaced by $\epsilon/2a$ and M by $2aM$, we infer that

$$|\nabla v(y)| \leq C(n)\left[\frac{2aK}{\epsilon} + 2aM\log\frac{1}{d(y,\partial D)}\right] \ ,$$

provided that y lies on the radial line between q and the origin, and $0 < d(y,\partial D) < C(n,\epsilon/2a)$. Setting $x = 2ay$, we have $d(x,\partial B) = (2a)d(y,\partial D)$, and

$$2a\,|\nabla u(x)| \leq C(n)\left[\frac{2aK}{\epsilon} + 2aM\log\frac{2a}{d(x,\partial B)}\right] \ ,$$

provided that $0 < d(x,\partial B) < 2aC(n,\epsilon/2a)$. Division by $2a$ gives (2.3), for $0 < d(x,\partial B) < C(n,\epsilon,a)$ and x on the radial line from the center of B to p. □

In order to pass from balls to more general domains we will employ a method of Kellogg [7] featuring certain superharmonic dominating functions. For x in $\mathbb{R}^n$ we set $r(x) := |x|$ and $\theta(x) := \arccos\left(x_n/|x|\right)$. For $\alpha \in (0,\pi/2)$ we let $\mathcal{C}_\alpha$ be the open set

$$\mathcal{C}_\alpha := \{x \in \mathbb{R}^n : |x| > 0 \text{ and } \theta(x) < \alpha + \pi/2\} \ .$$

Lemma 3. *For any $\lambda \in (1/2,1)$ there exists $\alpha \in (0,\pi/2)$ and a corresponding function w of the form $w(x) = r^\lambda f(\cos\theta)$ which is continuous on $\overline{\mathcal{C}_\alpha}$, positive on $\overline{\mathcal{C}_\alpha}\backslash\{0\}$, and of class C^2 in $\mathcal{C}_\alpha$ with $\Delta w \leq 0$ there.*

Proof. The existence of such a barrier function, and of more general barrier functions, is discussed in section 4 of [6]. □

With these preliminaries, we may now prove our main theorem.

Proof of Theorem 2 In view of the assumed smoothness of $\partial\Omega$, there exists a number a such that $0 < a \leq 1/2$ and, at any point p of $\partial\Omega$, the two closed balls of radius a tangent internally and externally to $\partial\Omega$ at p lie entirely inside Ω and outside $\overline{\Omega}$, respectively, except at the point p. We consider an arbitrary point p on $\partial\Omega$, and apply a rigid motion so that $p = 0$ and $\partial\Omega$ near 0 is represented by the graph, $x_n = f(x_1,\ldots,x_{n-1})$, of a C^2

function f with $\nabla f(0) = 0$ and with Ω near 0 lying above the graph of f. Since the hypotheses and conclusions of the theorem are unaffected by perturbing φ with an additive constant, we may assume $\varphi(0) = 0$ and hence, by the Lipschitz condition,

$$|\varphi(z)| \le M\,|z| \le M d(\Omega) \ \ , \text{ for } z \in \partial\Omega \ . \tag{2.6}$$

Fix $\lambda \in (1/2, 1)$, and let $\mathcal{C}_\alpha$ and w denote the corresponding set and function described in Lemma 3. Then there exist positive constants $C_1(n)$ and $C_2(n)$ such that

$$C_1(n)\,|x|^\lambda \le w(x) \le C_2(n)\,|x|^\lambda \quad \text{for all x in } \overline{\mathcal{C}_\alpha} \ . \tag{2.7}$$

Moreover, a geometric argument shows that, if $|x| \le \delta := 2a\sin\alpha$ and $x \notin \overline{\mathcal{C}_\alpha}$, then x lies in the open ball of radius a tangent externally to $\partial\Omega$ at 0, and hence outside $\overline{\Omega}$; consequently, if $x \in \overline{\Omega}$ and $|x| \le \delta$, then $x \in \overline{\mathcal{C}_\alpha}$. Note that $0 < \delta < 2a \le 1$. If $z \in \partial\Omega$ and $|z| \le \delta$, then (2.6) and (2.7) yield

$$|u(z)| = |\varphi(z)| \le M\,|z| \le M\,|z|^\lambda \le \frac{M}{C_1(n)}\,w(z) \ .$$

On the other hand, if $x \in \overline{\Omega}$ and $|x| = \delta$, then (2.6), (2.7), and the maximum principle give

$$|u(x)| \le \max_{\partial\Omega}|\varphi| \le M d(\Omega) \le \frac{M d(\Omega)}{C_1(n)\delta^\lambda}\,w(x) \ .$$

Combining the last two inequalities and (2.7), we apply the maximum principle for subharmonic functions in the intersection of Ω with the open ball of radius δ about the origin to conclude that, for some positive constant $C(n,\Omega)$,

$$|u(x)| \le C(n,\Omega) M w(x) \le C(n,\Omega) M\,|x|^\lambda \quad , \text{ if } x \in \overline{\Omega} \text{ and } \ |x| \le \delta \ . \tag{2.8}$$

Next let B denote the ball of radius a internally tangent to Ω at 0. Near 0, ∂B is represented by the graph of a C^2 function $x_n = g(x_1, \ldots, x_{n-1})$, satisfying $|g(y)| \le |y|^2/a$ for $|y| < a$. By the smoothness and compactness of $\partial\Omega$, and the fact that $\nabla f(0) = 0$, there are positive constants η and D, depending only on Ω and not on the point p originally chosen, such that $|f(y)| \le D\,|y|^2 \le |y|$ for $|y| \le \eta$. We may stipulate also that $\eta < \delta/2\,(< a \le 1/2)$. Let $z \in \partial B$ with $|z| < \eta$, and set $z = (y, g(y))$, $\zeta = (y, f(y))$, where $|y| \le |z| < \eta$. Then

$$|u(\zeta)| = |\varphi(\zeta)| \le M\,|\zeta| \le M\,(|y| + |f(y)|) \le 2M\,|y| \le 2M\,|z| \ .$$

Also,

$$|z - \zeta| = |g(y) - f(y)| \le |g(y)| + |f(y)| \le |y|^2/a + |y| \le 2\,|y| < 2\eta < \delta \ .$$

We apply (2.8) translated to the boundary point ζ (instead of 0), and get

$$
\begin{aligned}
|u(z) - u(\zeta)| &\le C(n,\Omega)M\,|z-\zeta|^{\lambda} = C(n,\Omega)M\,|g(y)-f(y)|^{\lambda} \\
&\le C(n,\Omega)M\,(|g(y)|+|f(y)|)^{\lambda} \le C(n,\Omega)M\left(|y|^2/a + D\,|y|^2\right)^{\lambda} \\
&\le C(n,\Omega)M\left(a^{-1}+D\right)^{\lambda}|y|^{2\lambda} \le C(n,\Omega)M\,|z|^{2\lambda} \quad .
\end{aligned}
$$

But $2\lambda > 1$ while $|z| < 1$; thus by the triangle inequality we have

$$
|u(z)| \le |u(\zeta)| + |u(z)-u(\zeta)| \le C(n,\Omega)M\,|z| \quad ,
$$

valid for $z \in \partial B$ with $|z| < \eta$.

For $z \in \partial B$ we have also $|u(z)| \le \max|\varphi| \le Md(\Omega)$. We apply Lemma 2 to the restriction of u to B, with K replaced by $Md(\Omega)$, M by $C(n,\Omega)M$, and ϵ by $\eta = \eta(\Omega)$, and conclude that

$$
\begin{aligned}
|\nabla u(x)| &\le C(n)\left[\frac{Md(\Omega)}{\eta(\Omega)} + C(n,\Omega)M\log\frac{2a}{d(x,\partial B)}\right] \\
&\le C(n,\Omega)M\log\frac{d(\Omega)}{d(x,\partial B)} \quad ,
\end{aligned}
$$

provided that x lies on the radial line from 0 to the center of B and $0 < d(x,\partial B) \le \epsilon$ for some sufficiently small ϵ depending on n and Ω.

Now, for arbitrary x in Ω with $d(x,\partial\Omega) \le \epsilon$, there is a ball B centered at x and touching $\partial\Omega$ only on ∂B. We may adjust B, if necessary, so that its radius is a and x lies on the normal line to $\partial\Omega$, between the point of tangency and the center of B. Then $d(x,\partial B) = d(x,\partial\Omega)$, and we have verified (1.2) for such x.

Next suppose $x \in \Omega$ but $d(x,\partial\Omega) \ge \epsilon = \epsilon(n,\Omega)$. By a well known gradient estimate for harmonic functions (see, for example, section 2.7 of [3]),

$$
|\nabla u(x)| \le \frac{C(n)\max|u|}{d(x,\partial\Omega)} \quad .
$$

But $\max|u| \le Md(\Omega)$, while $2 \le d(\Omega)/d(x,\partial\Omega) \le d(\Omega)/\epsilon$; therefore,

$$
|\nabla u(x)| \le C(n)M\frac{d(\Omega)}{d(x,\partial\Omega)} \le C(n,\Omega)M\log\frac{d(\Omega)}{d(x,\partial\Omega)} \quad ,
$$

and (1.2) is verified for all x in Ω.

We give an example showing that one cannot relax too much in Theorem 2 the smoothness properties of $\partial\Omega$; indeed, if we assume the same boundary regularity as we do on the boundary function, the conclusion of Theorem 2 is no longer valid.

Recall that a *Lipschitz domain* in $\mathbb{R}^n$ is a domain whose boundary is locally representable by graphs of Lipschitz continuous functions in $n-1$ variables.

Example 1. *For any $n \geq 2$ there exists a bounded Lipschitz domain Ω in $\mathbb{R}^n$ and a Lipschitz continuous function φ on $\partial\Omega$ for which the solution to the Dirichlet problem, $u \in C(\overline{\Omega})$, $\Delta u = 0$ in Ω, $u = \varphi$ on $\partial\Omega$, does not satisfy $|\nabla u(x)| = O(\log 1/d(x, \partial\Omega))$ as $d(x, \partial\Omega) \to 0$.*

First we produce a domain Ω in the plane having the properties described. It is easily checked that in the right half plane the function

$$w(x, y) = \arctan \frac{y-2}{x} \quad ,$$

a branch of $\arg(z - 2i)$, is harmonic, with

$$w(0, y) = \begin{cases} \pi/2 , & \text{if } y > 2 , \\ -\pi/2 , & \text{if } y < 2 , \end{cases} \quad , \quad w_x(x, 0) = \frac{2}{4 + x^2} \quad .$$

Fix α with $1 < \alpha < 2$, and consider the Lipschitz domain

$$\Omega = \left\{ z = re^{i\theta} : 0 < r < 1 \text{ and } |\theta| < \pi\alpha/2 \right\} \quad ,$$

and the function u defined in Ω according to

$$u(z) = w\left(z^{1/\alpha}\right) \quad , \quad w(z) = u\left(z^{\alpha}\right) \quad .$$

The mapping $z \longrightarrow z^{\alpha}$ maps the right half of the open unit disk conformally onto Ω, and it follows that u is harmonic in Ω and Lipschitz continuous up to $\partial\Omega$. (In fact, $u \equiv -\pi/2$ on the straight portions of $\partial\Omega$, and u can be extended as a C^{∞} function in a neighborhood of the curved portion of $\partial\Omega$.) Observe that $u(x, 0) = w(t, 0)$ where $t = x^{1/\alpha}$, and for $0 < x \leq 1$,

$$u_x(x, 0) = w_t(t, 0)\frac{1}{\alpha}x^{(1/\alpha)-1} = \frac{2}{4 + x^{2/\alpha}}\, x^{(1/\alpha)-1} \geq \frac{2}{5}\, x^{(1/\alpha)-1} \quad .$$

But $d((x, 0), \partial\Omega) = x$ for $0 < x < 1/2$, so $|\nabla u(z)| \neq O(\log 1/d(z, \partial\Omega))$ in Ω.

It is now easy to construct an analogous example in $\mathbb{R}^n$ when $n > 2$. Let Ω be our domain in $\mathbb{R}^2$, and in $\mathbb{R}^n$ let Ω^n be the bounded Lipschitz domain

$$\Omega^n = \Omega \times (-1, 1)^{n-2} = \Omega \times (-1, 1) \times (-1, 1) \times \ldots \times (-1, 1) \quad .$$

Define also $v(x_1, \ldots, x_n) = u(x_1, x_2)$. Then v is harmonic in Ω^n with Lipschitz boundary values, but with

$$|\nabla v(x_1, 0, \ldots, 0)| = |\nabla u(x_1, 0| \neq O\left(\log \frac{1}{x_1}\right) \text{ as } x_1 \to 0 \quad .$$

3. Domains in $\mathbb{R}^2$

In certain planar domains we may use results from complex analysis to relax somewhat the C^2 requirement on the boundary. We will say that a bounded Jordan domain Ω in the plane is *Dini smooth* provided it has a parametrization $w : \partial D \longrightarrow \partial\Omega$ (D = unit disk), with Dini continuous derivative w' never vanishing on ∂D. (Thus, if

$$\omega(t) := \sup\left\{ \left|w'(\theta_1) - w'(\theta_2)\right| : e^{i\theta_1}, e^{i\theta_2} \in D,\ |\theta_1 - \theta_2| \leq t \right\} \quad ,$$

then

$$\int_0^1 \frac{\omega(t)}{t}\,dt < \infty \quad .)$$

Theorem 3. *Let Ω be a bounded Dini smooth Jordan domain in the plane, let $\varphi : \partial\Omega \longrightarrow R$ be Lipschitz continuous with Lipschitz constant M, and let u solve the Dirichlet problem $u \in C(\overline{\Omega})$, $\Delta u = 0$ in Ω, $u = \varphi$ on $\partial\Omega$. Then there exists a positive constant $C(\Omega)$ such that, for all z in Ω,*

$$|\nabla u(z)| \le C(\Omega) M \log \frac{d(\Omega)}{d(z, \partial\Omega)} \quad . \tag{3.1}$$

Proof. Let f map Ω conformally onto the upper half plane $\mathcal{H}$. It is well known, since Ω is a Jordan domain, that f extends to a homeomorphism of the closures, $f : \overline{\Omega} \longrightarrow \overline{\mathcal{H}}$, and furthermore, since Ω is Dini smooth, f' extends continuously to $\overline{\Omega}$ where it is never zero. (See [8], Theorem 3.5, page 48.) It follows that $v := u \circ f^{-1}$ is a bounded harmonic function on $\mathcal{H}$, continuous on $\overline{\mathcal{H}}$, and has boundary data $v = \psi := \varphi \circ f^{-1}$ on $\partial\mathcal{H}$. Since the derivative $\left(f^{-1}\right)'$ is uniformly bounded, for ψ on $\partial\mathcal{H}$ we have the Lipschitz condition

$$\begin{aligned} |\psi(x_1) - \psi(x_2)| &= \left|\varphi\left[f^{-1}(x_1)\right] - \varphi\left[f^{-1}(x_2)\right]\right| \le M \left|f^{-1}(x_1) - f^{-1}(x_2)\right| \\ &\le M \left\|\left(f^{-1}\right)'\right\|_\infty |x_1 - x_2| = M C(\Omega) |x_1 - x_2| \quad . \end{aligned}$$

Subtracting a constant from φ if necessary, we may assume that φ vanishes somewhere on $\partial\Omega$; then $|\varphi| \le M d(\Omega)$ on $\partial\Omega$, implying that also $|\psi| \le M d(\Omega)$ on $\partial\mathcal{H}$, $|u| \le M d(\Omega)$ on $\overline{\Omega}$, and $|v| \le M d(\Omega)$ on $\overline{\mathcal{H}}$. Next, noting that the result of Lemma 1 is translation invariant, we apply Lemma 1 with K replaced by $M d(\Omega)$, M by $M C(\Omega)$, ϵ by 1, and conclude that there are positive constants C and δ such that, for $z \in \mathcal{H}$ and $d(z, \partial\mathcal{H}) < \delta$,

$$|\nabla v(z)| \le C \left[M d(\Omega) + M C(\Omega) \log \frac{1}{d(z, \partial\mathcal{H})} \right] \quad .$$

We may assume $\delta \le 1/e$, so that this inequality simplifies to

$$|\nabla v(z)| \le C(\Omega) M \log \frac{1}{d(z, \partial\mathcal{H})} \quad , \text{ if } d(z, \partial\mathcal{H}) < \delta \; . \tag{3.2}$$

Let $m = m(\Omega)$ be a number such that $0 < m < 1$ and $m \le |f'| \le 1/m$ on $\overline{\Omega}$. It is a consequence of the Koebe distortion theory (see Corollary 1.4 on page 9 of [8]) that, for all z in Ω,

$$\frac{m}{16} \le \frac{d\left(f(z), \partial\mathcal{H}\right)}{d(z, \partial\Omega)} \le \frac{16}{m} \quad . \tag{3.3}$$

Set $\epsilon = m\delta/16$. If $z \in \Omega$ and $d(z, \partial\Omega) < \epsilon$, then (3.3) gives $d\left(f(z), \partial\mathcal{H}\right) < \delta$, and then the chain rule, (3.2), and (3.3) lead to

$$\begin{aligned} |\nabla u(z)| &= |\nabla v\left(f(z)\right)| \left|f'(z)\right| \leq \frac{1}{m} |\nabla v\left(f(z)\right)| \\ &\leq \frac{C(\Omega)M}{m} \log \frac{1}{d(f(z), \partial\mathcal{H})} \leq \frac{C(\Omega)M}{m} \log \frac{16}{md(z, \partial\Omega)} \quad . \end{aligned}$$

But $d(\Omega)/d(z, \partial\Omega) \geq 2$, and so, in the case $d(z, \partial\Omega) < \epsilon$, (3.1) follows for some suitable new constant $C(\Omega)$. The case $d(z, \partial\Omega) \geq \epsilon$ is handled exactly as in the proof of Theorem 2. □

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G. N. Hile, Mathematics Department, University of Hawaii, Honolulu, HI, USA 96822

E-mail address: `hile@math.hawaii.edu.`

A. Stanoyevitch, Mathematics Department, University of Guam, UOG Station, Mangilao, GU 96923

E-mail address: `alexs@uog9.uog.edu`