POINCARÉ INEQUALITIES AND STEINER SYMMETRIZATION

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Abstract. A complete geometric characterization for a general Steiner symmetric domain $\Omega \subset \mathbb{R}^n$ to satisfy the Poincaré inequality with exponent $p > n-1$ is obtained and it is shown that this range of exponents is best possible. In the case where the Steiner symmetric domain is determined by revolving the graph of a Lipschitz continuous function, it is shown that the preceding characterization works for all $p > 1$ and furthermore for such domains a geometric characterization for a more general Sobolev–Poincaré inequality to hold is given. Although the operation of Steiner symmetrization need not always preserve a Poincaré inequality, a general class of domains is given for which Poincaré inequalities are preserved under this operation.

SECTION 1: INTRODUCTION

Let Ω be a domain in $\mathbb{R}^n$ $(n \geq 2)$ with finite volume: $m_n(\Omega) < \infty$. Given an integrable function u on Ω, we let u_Ω denote its average value on Ω, *i.e.,*

$$u_\Omega = \fint_\Omega u(x)\, dx.$$

For each number p, $1 \leq p < \infty$, the domain Ω is said to be a *p–Poincaré domain* provided that

$$M_p(\Omega) := \sup_u \frac{\|u - u_\Omega\|_{L^p(\Omega)}}{\|\nabla u\|_{L^p(\Omega)}} < \infty,$$

where the supremum is taken over all nonconstant functions u in the Sobolev space $W^{1,p}(\Omega)$. Thus p–Poincaré domains support the *p–Poincaré inequality*:

$$\int_\Omega |u - u_\Omega|^p\, dx \leq M \int_\Omega |\nabla u|^p\, dx.$$

By the density of smooth functions in $W^{1,p}(\Omega)$ ([19], [8]), the Poincaré inequality need only be checked for locally Lipschitz continuous functions. The Poincaré inequalities are prototypical examples of Sobolev inequalities which are extensively used in PDE and related fields, see [18], [1], [26], and Chapter 7 of [11]. The geometry of Poincaré domains is quite complicated and a complete geometric characterization remains an elusive unsolved problem, even for the case of a simply connected planar domain (see, however, [13]). Notice that there is not much hope

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for a general geometric characterization since for example the removal of a closed set of vanishing (n-1)-dimensional measure from a p–Poincaré domain results in a new p–Poincaré domain. We will mostly be dealing with the p–Poincaré inequalities when $p > 1$. The "isoperimetric" case $p = 1$ is more tractable, see [12], [18] and [25]. For information on Poincaré type inequalities in case $p < 1$, see [5]. We point out that iteration arguments can be used to show that certain local inequalities imply a corresponding global inequality in smooth domains and even in domains that satisfy a twisted interior cone condition (John domains)–see [3], [6], and [14]. For sufficiently "nice" domains (smooth or uniform, for example) a Poincaré inequality can also be shown by extending the functions to all of $\mathbb{R}^n$, see [7], [13] and the references therein.

In this paper we will give a geometric characterization of p–Poincaré domains where we restrict to the class of Steiner symmetric domains $\Omega \subset \mathbb{R}^n$. Our characterization will work only when $p > n-1$, and will depend on the Euclidean distance function:

$$\delta_\Omega(\overrightarrow{x}) = \mathrm{dist}[\overrightarrow{x}, \partial\,\Omega].$$

In order to formulate our results, it will be convenient to split the coordinates of a point $\overrightarrow{x} = (x_1, x_2, \cdots, x_n)$ as (x_1, x') where $x' = (x_2, x_3, \cdots, x_n) \in \mathbb{R}^{n-1}$. For each $t \in \mathbb{R}$ and any set $A \subset \mathbb{R}^n$ we define the *cross section* of A *at level* t as

$$A_t = \{x \in A : x_1 = t\}$$

and the *projection* of A onto the x_1-axis as

$$\mathrm{Proj}_{x_1}(A) = \{x \in \mathbb{R} : A_x \neq \emptyset\}.$$

The *Steiner symmetrization* (with respect to the x_1-axis) of the domain Ω is the domain

$$\Omega^\star = \left\{(x, x') \in \mathbb{R} \times \mathbb{R}^{n-1} : |x'|^{n-1} < \frac{1}{\omega_{n-1}} m_{n-1}(\Omega_x)\right\},$$

where ω_n denotes the volume of the unit ball in $\mathbb{R}^n$: $\omega_n = m_n(\mathrm{Ball}^n(1))$. The domain Ω is said to be a *Steiner symmetric domain* (with respect to the x_1-axis) if $\Omega = \Omega^\star$. We will also make use of the so-called k_p metric on Ω which is defined as follows:

$$k_p^\Omega(\overrightarrow{x}, \overrightarrow{y}) = \inf_\gamma \int_\gamma \frac{ds}{\delta_\Omega^{(n-1)/(p-1)}} \qquad (\overrightarrow{x}, \overrightarrow{y} \in \Omega),$$

where the infimum is taken over all rectifiable curves γ joining $\overrightarrow{x}$ to $\overrightarrow{y}$ inside Ω and the integration is with respect to arclength. The k_p metrics have been used by Gehring and Martio [10] and by Smith and Stegenga [23]. Observe that when $\Omega = \Omega^\star$ is a Steiner symmetric domain then the x_1-axis is a geodesic for k_p^Ω so that for $t_1, t_2 \in \mathrm{Proj}_{x_1}(\Omega)$ we have

$$k_p^\Omega((t_1, 0'), (t_2, 0')) = \int_{t_1}^{t_2} \frac{dt}{\delta_\Omega(t, 0')^{(n-1)/(p-1)}}.$$

Also, whenever $0 \in \mathrm{Proj}_{x_1}(\Omega)$ then for each $x \in \mathrm{Proj}_{x_1}(\Omega)$ we can define $T(\Omega_x)$ (here T stands for "tail") as a component of $\Omega \setminus \Omega_x$ which does not contain $(0, 0')$. Except when $x = 0$ there is only one such component. We are now ready to state the first of our three main results.

Theorem A. *(Geometric characterization of Steiner symmetric Poincaré domains) Let $\Omega \subseteq \mathbb{R}^n$ be a Steiner symmetric domain of finite volume. We assume*

$$\Omega = \left\{(x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1} : |x'| < \varphi(x_1)\right\} = \Omega^*$$

with $|\varphi| \leq M < \infty$. We may assume $\varphi(0) > 0$. Let $n-1 < p < \infty$. Then Ω is a p–Poincaré domain if and only if

$$\sup_{x \in \mathrm{Proj}_{x_1} \Omega} k_p^{\Omega}\left((0,0'),(x,0')\right)^{p-1} m_n\left(T(\Omega_x)\right) < \infty. \tag{1}$$

Furthermore, if $p = n-1$ and Ω is a p–Poincaré domain then (1) remains valid. Finally, for each $p \in (1, n-1]$ there is a Steiner symmetric domain $\Omega \subset \mathbb{R}^n$ of finite volume which is not a p–Poincaré domain but for which (1) *is valid.*

The operation of Steiner symmetrization is a natural one for many problems in PDE. For connections of the Steiner symmetrization as well as of types of symmetrizations with such problems, three good references are [2], [16] and (the classical) [21]. Each of these contains an extensive bibliography. In particular, when $p = 2$, the p–Poincaré constant $M_2(\Omega)$ is the reciprocal of the square root of the smallest positive eigenvalue for the Laplace operator with Neumann boundary conditions on Ω–see [8], [17], §4.10 of [18] and §4 of [24] for more on this connection. In 1948, Pólya proved [20] that the smallest positive eigenvalue for Laplace's operator with Dirichlet boundary conditions on Ω will never decrease under Steiner symmetrization. The corresponding result is no longer true if Neumann boundary conditions are to replace those of Dirichlet. In fact there exists a domain $\Omega \subset \mathbb{R}^2$ which is a p–Poincaré domain for all p but whose Steiner symmetrization $\Omega^\star$ fails to be a p–Poincaré domain for any p (see Example 6.10 in [22]). Our next result is a direct extension from two to any number of dimensions of one of the main results in a recent paper by Smith, Stegenga, and the second author (see Theorem C of [22]). It gives a class of domains for which the Poincaré inequalities are preserved under the operation of Steiner symmetrization.

Theorem B. *(Steiner symmetrization preserves Poincaré inequalities). Let $\Omega \subseteq \mathbb{R}^n$ be a domain satisfying*

$$\Omega_x = \{x\} \times \mathrm{Ball}^{n-1}\left(\overrightarrow{\gamma(x)}, \varphi(x)\right) \qquad (x \in \mathbb{R})$$

where $\overrightarrow{\gamma} : \mathbb{R} \longrightarrow \mathbb{R}^{n-1}$ and $\varphi : \mathbb{R} \longrightarrow [0, M]$ $(M < \infty)$. If Ω is a p–Poincaré domain with $n-1 < p < \infty$ then so is its Steiner symmetrization Ω^*.

Finally, we give a more restricted class of Steiner symmetric domains for which the characterization of Theorem A remains valid for all $p > 1$. These domains Ω will be obtained by revolving the graph of a Lipschitz continuous function $\Phi : \mathbb{R} \longrightarrow [0, \infty)$ about the x_1-axis. In fact, for such domains, we obtain the following geometric characterization of a more general Sobolev-Poincaré inequality.

Theorem C. *Assume that $\Phi : \mathbb{R} \longrightarrow [0, \infty)$ is Lipschitz continuous and*

$$\Omega = \left\{(x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1} : |x'| < \Phi(x_1)\right\} = \Omega^*$$

is a domain of finite volume. We may assume $\Phi(0) > 0$. Let p and q be positive numbers satisfying $1 < p \le q$. If $p < n$ we assume also that $q \le \frac{np}{n-p}$. The following are equivalent:

(i) *There exists a positive number C such that the inequality*

$$\|u - u_\Omega\|_{L^q(\Omega)} \le C\|\nabla u\|_{L^p(\Omega)}$$

holds for all Sobolev functions $u \in W^{1,p}(\Omega)$.

(ii)

$$\sup_{x \in \mathrm{Proj}_{x_1}(\Omega)} \left(\int_{\min\{0,x\}}^{\max\{0,x\}} |\Phi(s)|^{(1-n)/(p-1)}\, ds \right)^{p-1} m_n(T(\Omega_x))^{\frac{p}{q}} < \infty$$

Note that since Φ is Lipschitz, $\Phi(s)$ and $\delta_\Omega(s, 0')$ are always comparable, so the quantity in the above supremum is comparable to the corresponding quantity in Theorem A.

This paper is organized as follows. In Section 2 we formulate some preliminary lemmas which will be needed in the proofs of the principal results. Section 3 gives the proofs of Theorems A and B. In the final Section 4 we prove Theorem C and also construct a simple example to show that Theorem A cannot in general remain valid if the p–Poincaré inequality is replaced by the more general one considered in Theorem C (with $q > p$).

We invoke the customary conventions regarding constants. The same symbol for a constant may take on different values at different occurences. If we wish to stress that a constant C depends only on certain parameters, say p and n, we write $C = C(n, p)$. The notation $C \lesssim D$ shall usually indicate that C is dominated by an absolute constant A times C ($C \le AD$), although in some proofs for convenience we may allow A to depend on certain parameters if it is well understood that D may depend on these parameters as well. The notation $C \approx D$ is equivalent to $C \lesssim D$ and $D \lesssim C$. For example, if $|t| < \pi/2$ then $\sin t \approx t$.

SECTION 2: LEMMAS

Here we gather an assortment of results needed to prove the main theorems. We begin with the following Sobolev-type embedding theorem which is a consequence of inequalities (7.34) and (7.41) in [11]. See also Lemma 1.7 of [4].

Lemma 2.1. *If $B \subseteq \mathbb{R}^n$ is a ball of radius R, $p > n$, and $u \in W^{1,p}(B)$, then*

$$|u(x_1) - u(x_2)| \le C(n,p)|x_1 - x_2|^{1-\frac{n}{p}} \left(\int_B |\nabla u|^p dx \right)^{\frac{1}{p}},$$

for all x_1 and x_2 in B.

Our next two results provide additional formulations of the p–Poincaré inequality. In formulating the first result it will be convenient to introduce the following class of functions.

Notation. *For a domain $\Omega \subseteq \mathbb{R}^n$, we let $\mathrm{Lip}_{loc}(\Omega)$ denote the class of functions on Ω which are locally Lipschitz continuous on Ω.*

Lemma 2.2. *Let $\Omega \subseteq \mathbb{R}^n$ be a domain of finite volume, $B_0 = \mathrm{Ball}^n(x_0, r_0) \subset \Omega$ with $r_0 \le \frac{1}{2}\delta_\Omega(x_0)$. Letting*

$$M^p_{p,\Omega}(B_0) = \sup_u \left\{ \frac{\int_\Omega |u|^p dx}{\int_\Omega |\nabla u|^p dx} : u \in (\mathrm{Lip}_{loc} \cap W^{1,p})(\Omega) \setminus \{0\} \text{ and } u = 0 \text{ on } B_0 \right\}$$

$$N^p_{p,\Omega}(B_0) = \sup_u \left\{ \frac{m_n(u=1)}{\int_\Omega |\nabla u|^p dx} : u \in (\mathrm{Lip}_{loc} \cap W^{1,p})(\Omega) \setminus \{0\},\, 0 \le u \le 1, u = 0 \text{ on } B_0 \right\}$$

we have $M^p_p(\Omega) \le c(n,p) M^p_{p,\Omega}(B_0)$ and $M^p_{p,\Omega}(B_0) \le c(p) N^p_{p,\Omega}(B_0) \lesssim \left(\frac{m_n(\Omega)}{m_n(B_0)}\right) M^p_p(\Omega)$.

The proof is accomplished by a series of "truncation" arguments of the type often used in PDE. A good general reference for such material is the treatise by Maz'ja [18], where the results combined in Lemma 2.2 can be found. For the convenience of the reader we briefly sketch the ideas involved in the proof.

The first inequality of the Lemma follows by decomposing $v = u - u_{(3/2)B_0}$ as $v = \psi v + (1 - \psi v)$ where ψ is an appropriate cut-off function and then applying the Sobolev and Poincaré inequalities for ψv and finally using Lemma 2.3 below.

For the second inequality, let $u \in \mathrm{Lip}_{loc}(\Omega) \cap W^{1,p}(\Omega) \setminus \{0\}$ with $u = 0$ in B_0. We must show that $R_M(u) := \int_\Omega |u|^p \, dx / \int_\Omega |\nabla u|^p \, dx \lesssim N_{p,\Omega}(B_0)$.

For $j \in \mathbb{Z}$, define

$$A_j = \left\{ x \in \Omega : 2^{j-1} \le |u(x)| < 2^j \right\}.$$

We may write

$$\int_\Omega |u|^p \, dx \approx \sum_{j \in \mathbb{Z}} 2^{jp} m_n(A_j). \tag{2}$$

Define for $j \in \mathbb{Z}$

$$v_j(x) = \max \left\{ 0, \min \left\{ 1, \frac{|u(x)| - 2^{j-1}}{2^{j-1}} \right\} \right\}.$$

Differentiating, yields

$$|\nabla v_j(x)| = \begin{cases} |\nabla u(x)| 2^{-j+1} & \text{if } x \in A_j \\ 0 & \text{if } x \notin A_j \end{cases} \tag{3}$$

Note that for each j, $v_j \in (\mathrm{Lip}_{loc} \cap W^{1,p})(\Omega) \setminus \{0\}$, $0 \le v_j \le 1$ and $v_j = 0$ on B_0. Whence $m_n(\{v_j = 1\}) \le N^p_{p,\Omega}(B_0) \int\limits_\Omega |\nabla v_j|^p \, dx$. Using this inequality and then (3) we obtain $m_n(A_j) \le m_n(\{v_{j-1} = 1\}) \le N^p_{p,\Omega}(B_0) \int\limits_\Omega |\nabla v_{j-1}|^p dx \le N^p_{p,\Omega}(B_0) 2^{(-j+2)p} \int\limits_{A_j} |\nabla u|^p dx$.

Summing up by using (2) gives

$$\int_\Omega |u|^p dx \le c(p) N^p_{p,\Omega}(B_0) \int_\Omega |\nabla u|^p \, dx$$

as desired.

The last inequality follows from Lemma 2.3 below.

Lemma 2.3. *Let $\Omega \subseteq \mathbb{R}^n$ be a p–Poincaré domain, $1 \le p < \infty$ and $A \subset \Omega$ be any measurable subset of positive volume. For $u \in W^{1,p}(\Omega)$, we have*

$$\int_\Omega |u - u_A|^p dx \le c(p) \frac{m_n(\Omega)}{m_n(A)} K_p^p(\Omega) \int_\Omega |\nabla u|^p dx \,.$$

The proof of Lemma 2.3 is accomplished by adding and subtracting u_Ω from $u - u_A$ and then using the triangle inequality and Hölder's inequality. We omit the details.

In order to deal effectively with Steiner symmetric domains $\Omega = \Omega^*$, we will need to "discretize" the function $\delta_\Omega(x, 0') \quad (x \in \mathbb{R}, \quad 0' \in \mathbb{R}^{n-1})$ along the central axis. The following result, whose proof relies on a Whitney type decomposition argument, will accomplish this for us.

Lemma 2.4. *Let $\Omega = \Omega^* \subseteq \mathbb{R}^n$ be a Steiner symmetric domain. Write $(a_\Omega, b_\Omega) = \operatorname{Proj}_{x_1} \Omega$. There exists a sequence $\langle a_i \rangle_{i \in I}$ with $I \subseteq \mathbb{Z}$ an interval, such that*

$$a_\Omega \le a_i < a_{i+1} \le b_\Omega$$

for each $i \in I \cap (I - 1)$, and for each $x \in (a_\Omega, b_\Omega)$, we have

$$\min_{i \in I} \operatorname{dist}[(x, 0'), [\partial\Omega]_{a_i}] \le 2\delta_\Omega(x, 0') \,.$$

Furthermore, if $a_{i-1} \le x \le a_i$ we have

$$\min_{j \in \{i-1, i\}} \operatorname{dist}\left[(x, 0'), [\partial\Omega]_{a_j}\right] \le 2\delta_\Omega(x, 0')$$

provided $i \notin \{0, 1\}$.

Finally, for any given $x_0 \in \operatorname{Proj}_{x_1} \Omega$, the construction can be made in such a way that $\delta_\Omega(x_0, 0') = \operatorname{dist}[(x_0, 0'), [\partial\Omega]_{a_0}]$.

Proof. Fix $x_0 \in (a_\Omega, b_\Omega)$. For brevity we write $\delta_\Omega(x)$ for $\delta_\Omega(x, 0')$.

Choose $a_0 \in [a_\Omega, b_\Omega]$ satisfying

$$\operatorname{dist}[(x_0, 0'), [\partial\Omega]_{a_0}] = \delta_\Omega(x_0) =: \delta_0 \,.$$

Observe that if $0 < \Delta x < \delta_\Omega(x)$, the triangle inequality implies $\operatorname{dist}[(x_0 + \Delta x, 0'), [\partial\Omega]_{a_0}] \le \delta_0 + \Delta x$ and $\delta_\Omega(x_0 + \Delta x) \ge \delta_0 - \Delta x$ whence

$$\frac{\operatorname{dist}\left[(x_0 + \Delta x, 0'), [\partial\Omega]_{a_0}\right]}{\delta_\Omega(x_0 + \Delta x)} \le \frac{\delta_0 + \Delta x}{\delta_0 - \Delta x}$$

so if $\Delta x \le (\frac{1}{3})\delta_\Omega(x_0)$ it follows that

$$\operatorname{dist}\left[(x_0 + \Delta x, 0'), [\partial\,\Omega]_{a_0}\right] \le 2\delta_\Omega(x_0 + \Delta x) \,. \tag{4}$$

Let $\Delta_0 = \inf\{\Delta x : (4) \text{ is false}\}$ and put $x_1 = x_0 + \Delta_0$. Note by the above comment, $\Delta_0 \ge \frac{1}{3}\delta_\Omega(x_0)$. If $x_1 = b_\Omega$ we need not construct any more a_i's $(i > 0)$. Otherwise, we choose $a_1 \in [a_\Omega, b_\Omega]$ satisfying

$$\operatorname{dist}\left[(x_1, 0'), [\partial\,\Omega]_{a_1}\right] = \delta_\Omega(x_1) =: \delta_1 \,.$$

The following claim shows that we must have $a_1 > a_0$.

Sublemma. *If $a_j < a_i$ and $(x, 0')$ is closer to $[\partial\,\Omega]_{a_j}$ than to $[\partial\,\Omega]_{a_i}$ then so is $(t, 0')$ for all $t < x$.*

(The proof of the sublemma is an exercise in elementary plane geometry, we omit the details.) Indeed, if it were to happen that $a_1 < a_0$ then by the sublemma and the properties of a_0 and a_1, we would have

$$\operatorname{dist}\big[(x_0, 0'), [\partial\,\Omega]_{a_1}\big] < \operatorname{dist}\big[(x_1, 0'), [\partial\,\Omega]_{a_0}\big] = \delta_\Omega(x_0)$$

which is impossible, so indeed $a_1 > a_0$.

We put

$$\Delta_1 = \inf\big\{\Delta x : \operatorname{dist}\big[(x_1 + \Delta x, 0'), [\partial\,\Omega]_{a_1}\big] > 2\delta_\Omega(x_1 + \Delta x)\big\}$$

and $x_2 = x_1 + \Delta_1$. If $x_2 = b_\Omega$ stop this construction, otherwise we proceed as before to seek to obtain $a_2 > a_1$ having the same relationship to x_2 as a_1 did to x_1. We continue this construction iteratively (and perhaps indefinitely) in the same fashion to construct a desired sequence $\langle a_i \rangle_{i \geq 0}$. The fact that at each step, $\Delta_k > \frac{1}{3}\delta_\Omega(x_k)$ guarantees that $x_k \nearrow b_\Omega$. We can similarly construct a sequence $\langle a_i \rangle_{i<0}$. By combining these two sequences, we obtain a sequence $\langle a_i \rangle_{i \in I}$, which by virtue of the construction, clearly satisfies all of the desired properties except for the last inequality.

We now establish the last inequality of Lemma 2.4. By symmetry we may assume that $i > 1$. Fix $\widetilde{x} \in [a_{i-1}, a_i]$, and choose $k \in I$ such that

$$\min_{j \in I} \operatorname{dist}\big[(\widetilde{x}, 0'), [\partial\Omega]_{a_j}\big]$$

is attained when $j = k$. We must show that $k \in \{i-1, i\}$, and we shall accomplish this by method of contradiction.

Case 1. $a_k < a_{i-1}$.

Regardless of where x_k is located, be it $x_k \leq a_k$ or $x_k > a_k$, it follows from the construction of the sequences $\langle a_j \rangle_{j \in I}$ and $\langle x_j \rangle_{j \in I}$ and simple plane geometry that $x_{k+1} > \widetilde{x}$.

We now have

$$a_k < a_{k+1} \leq a_{i-1} \leq \widetilde{x} < x_{k+1}. \tag{5}$$

Since (by choice of k) no points $(a_j, \varphi(a_j))$ can lie inside the (two dimensional) semi-circle with center $(x, y) = (\widetilde{x}, 0)$ and radius $\operatorname{dist}[(\widetilde{x}, 0'), [\partial\Omega]_{a_k}]$, we must have

$$\varphi(a_{k+1})^2 \geq (\widetilde{x} - a_k)^2 + \varphi(a_k)^2 - (\widetilde{x} - a_{k+1})^2. \tag{6}$$

Also by virtue of the construction, we can write

$$\operatorname{dist}\left[(x_{k+1}, 0'), (a_k, \varphi(a_k))\right]^2 = 4 \operatorname{dist}\left[(x_{k+1}, 0'), (a_{k+1}, \varphi(a_{k+1}))\right]^2. \tag{7}$$

If we estimate the right side of (7), using (6), then use the inequality: $(\widetilde{x} - a_k)^2 \geq (\widetilde{x} - a_{k+1})^2 + (a_{k+1} - a_k)^2$, and finally the triangle inequality we obtain a contradictory inequality. This shows that Case 1 cannot occur.

Case 2. $a_k > a_i$.

The treatment here is similar to that in Case 1. We omit the details. ∎

Lemma 2.5. *Let R be the cylinder defined by*

$$R = \big\{(x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1} : |x_1| < M \text{ and } |x'| < M\big\}.$$

If $p \geq n-1$ then

$$\sup_{0<b<M} k_p^R\big((0,0'),(b,0')\big)^{p-1} m_n\big(R \cap \{x_1 > b\}\big) \leq C(M,p,n).$$

Moreover if $p < n-1$ then this supremum is infinite

Remark. *The last statement of the Lemma shows that a principal result [Theorem A] in* [SmStSt–P] *used to prove another main result [Theorem C] therein which is the two dimensional analogue of our Theorem B here, cannot be generalized to $n > 2$ dimensions without assuming (at least) $p \geq n-1$.*

Proof. Define $\Phi(b) := k_p^R\big((0,0'),(b,0')\big)^{p-1} m_n\big(R \cap \{x_1 > b\}\big)$. We have $\Phi(b) = c(n)\big(\int_{M-b}^{M} \frac{ds}{s^{(n-1)/(p-1)}}\big)^{p-1}(M-b)M^{n-1}$. Now

$$\int_{M-b}^{M} \frac{ds}{s^{(n-1)/(p-1)}} = C(n,p)\begin{cases} \big|M^{1-\frac{n-1}{p-1}} - (M-b)^{1-\frac{n-1}{p-1}}\big|, & \text{if } n \neq p \\ \log \frac{1}{1-b/M}, & \text{if } n = p\,. \end{cases}$$

Case 1. $n = p$.

$$\begin{aligned} \Phi(b) &\leq C(M,p,n)\left(1 - \frac{b}{M}\right)\left[\log \frac{1}{1-b/M}\right]^{n-1} \\ &\leq C(M,p,n) \max_{0<t\leq M} t\left[\log \frac{1}{t}\right]^{n-1} = C(M,p,n) \end{aligned}$$

Case 2. $n \neq p$.

Let $\gamma = (p-1)\big(1 - \frac{n-1}{p-1}\big) + 1 = p+1-n$. We have $\Phi(b) \leq C(M,p,n) + C(M,p,n)(M-b)^{\gamma} \leq C(M,p,n)$ for all $b \in (0,M)$ if and only if $\gamma \geq 0$, *i.e.*, $p \geq n-1$. ∎

SECTION 3: PROOFS OF MAIN RESULTS

In this section we shall prove Theorems A and B.

Strategy of Proofs. We first show sufficiency of the supremum (1) being finite for the p–Poincaré inequality to hold on Ω^*. Next we show that for any domain Ω satisfying the conditions of Theorem B (in particular, if $\Omega = \Omega^*$ is as in Theorem A), the supremum in (1) (for Ω^*) being infinite will cause the p–Poincaré inequality on Ω to fail ($n-1 < p < \infty$). This will establish the necessity part of Theorem A, and also (in light of the previously established sufficiency for Theorem A) at the same time prove Theorem B. We finish the proofs by giving counterexamples which demonstrate that the range of exponents coverered in Theorem A is sharp.

PART A: Condition (1) implies Ω^* is a p–Poincaré domain. ($n-1 < p < \infty$).

By Lemma 2.2 we need only check that if $u \in (\mathrm{Lip}_{loc} \cap W^{1,p})(\Omega)$, satisfies $0 \leq u \leq 1$ and $u \restriction B_0 = 0$ where $B_0 = \mathrm{Ball}^n\left(\overrightarrow{0}, \frac{1}{2}\delta_\Omega\big(\overrightarrow{0}\big)\right)$ then $m_n\big(\{u = 1\}\big) \leq N\int_\Omega |\nabla u|^p dx$ for some finite positive constant N.

For convenience we introduce some more notation. We write Ω for Ω^* in the remainder of this proof. For $s \geq 0$, $\Omega_s^+ = \big\{(x,x') \in \Omega : x > s\big\}$, for $s \leq 0$, $\Omega_s^- = \big\{(x,x') \in \Omega : x < s\big\}$. We write $A = \{u = 1\}$, and define

$$A_1 = A \cap \bigcup_{t \in T_1} \Omega_t \text{ where } T_1 = \left\{t : u(t,x') \leq \frac{1}{2} \text{ for some } x'\right\}$$
$$A_2 = A \cap \bigcup_{t \in T_2} \Omega_t \text{ where } T_2 = \left\{t : u(t,x') > \frac{1}{2} \text{ for all } x'\right\}.$$

Now, let $t \in T_1$ and assume that $\Omega_t \cap A \neq \emptyset$. Therefore $\operatorname{osc}_{\Omega_t} u \geq \frac{1}{2}$ so we can apply Lemma 2.1 to the restriction $u\restriction_{\Omega_t}$ for a.e. such t to conclude

$$\frac{1}{2} \leq C(M,p)\operatorname{diam}(\Omega_t)^{1-\frac{n-1}{p}} \left(\int_{\Omega_t} |\nabla u|^p d\mathcal{H}^{n-1}\right)^{\frac{1}{p}}.$$

Whence for such t we have

$$\int_{\Omega_t} |\nabla u|^p d\mathcal{H}^{n-1} \geq C(M,p)$$

and we conclude

$$\begin{aligned} m_n(A_1) &\lesssim \int_{\{t \in T_1 : A \cap \Omega_t \neq \emptyset\}} \varphi(t)^{n-1} dt \leq \int_{\{t \in T_1 : A \cap \Omega_t \neq \emptyset\}} M^{n-1} dt \\ &\leq C(M,p) \int_{\{t \in T_1 : A \cap \Omega_t \neq \emptyset\}} \int_{\Omega_t} |\nabla u|^p d\mathcal{H}^{n-1} dt \\ &\leq C(M,p) \int_{\Omega} |\nabla u|^p dx. \end{aligned}$$

We have left to obtain a similar bound for $m_n(A_2)$. We put

$$\begin{aligned} s_0^+ &= \inf\{s : 0 < s \in T_2 \text{ or } s = \infty\} \\ s_0^- &= \sup\{s : 0 > s \in T_2 \text{ or } s = -\infty\}. \end{aligned}$$

Certainly we have (letting $\Omega_\infty^+ = \Omega_{-\infty}^- = \emptyset$)

$$m_n(A_2) \leq m_n(\Omega_{s_0^+}^+) + m_n(\Omega_{s_0^-}^-).$$

We will prove that $m_n(\Omega_{s_0^+}^+) \leq N \int_\Omega |\nabla u|^p dx$. The corresponding inequality for $m_n(\Omega_{s_0^-}^-)$ is proved in the same fashion. Let $t \geq s_0^+$, $t \in T_2$. We must show that

$$m_n\big(\Omega_t^+\big) \leq N \int_\Omega |\nabla u|^p dx. \tag{8}$$

Case 1. $\Omega_t \cap \frac{3}{2}B_0 \neq \emptyset$.

Let $x' \in \mathrm{Ball}^{n-1}\left(0', \frac{1}{2}\delta_\Omega(0)\right)$. Then for a.e. such x',

$$\begin{aligned}\frac{1}{2} &\le \int_0^t \left|\frac{\partial}{\partial x_1} u(s,x')\right| ds \\ &\le \left(\int_0^t \left|\nabla u(s,x')\right|^p ds\right)^{\frac{1}{p}} t^{\frac{p-1}{p}} .\end{aligned}$$

Thus

$$\int_0^t |\nabla u(s,x')|^p ds \ge C(p)\delta_\Omega(0)^{1-p}$$

so that

$$\begin{aligned}\int_\Omega |\nabla u|^p dx &\ge \int_{\mathrm{Ball}^{n-1}(0',\frac{1}{2}\delta_\Omega(0))} \int_0^t \left|\nabla u(s,x')\right|^p ds \; d\mathcal{H}^{n-1}(x') \\ &\ge C(n,p)\delta_\Omega(0)^{n-p}\end{aligned}$$

and thus

$$\begin{aligned}m_n(\Omega_t^+) \le m_n(\Omega) &\le C(n,p)\frac{m_n(\Omega)}{m_n(B_0)}\delta_\Omega(0)^p \int_\Omega |\nabla u|^p dx \\ &\le C(n,p,M)\frac{m_n(\Omega)}{m_n(B_0)} \int_\Omega |\nabla u|^p dx \,.\end{aligned}$$

Case 2. $\Omega_t \cap \frac{3}{2}B_0 = \emptyset$.

It certainly would suffice to prove (8) with u being replaced by

$$\widetilde{u}(x) = \widetilde{u}(x_1,x') = \begin{cases} u(x) & \text{if } x_1 \le t \\ \max\{\frac{1}{2}, u(x)\} & \text{if } x_1 > t \,. \end{cases}$$

Because of this we may assume that $u \ge \frac{1}{2}$ throughout Ω_t^+. Let Q_0 be a largest possible cube of sidelength $\ell(Q_0) \in S := \{2^{-n} : n \in \mathbb{Z}\}$ which is centered on the x_1–axis and lies in B_0. Adjacent to the right face of Q_0 we construct another cube Q_1 which is also centered on the x_1–axis and whose side length $\ell(Q_1) \in S$ is as large as possible so that $2Q_1 \subset \Omega$. Next construct a cube Q_2 adjacent to the right face of Q_1 in the same fashion as Q_1 was constructed from Q_0. Continue in this way to construct a chain of cubes $\langle Q_i\rangle_{i=0}^N$ where $\ell(Q_i) \approx \delta_\Omega(x)$ for all $x \in Q_i$ such that $Q_N \subset \Omega_t^+$ but $Q_{N-1} \not\subset \Omega_t^+$. Now for each $i < N$, since $\ell(Q_i) \approx \ell(Q_{i+1})$, $Q_i \cup Q_{i+1}$ is a dilation of one of a finite collection of p–Poincaré domains consisting of a unit cube with a smaller cube attached to and centered on one face (any domain G with boundary ∂G of class $\mathcal{C}$ is a p–Poincaré domain for each $p \in [1,\infty)$ – see for example [9] Theorem V.4.19). Therefore $M_p(Q_i \cup Q_{i+1}) \approx \ell(Q_i)$ and by invoking Lemma

2.3 we obtain
(9)

$$
\begin{aligned}
\frac{1}{2} \le |u_{Q_0} - u_{Q_N}| &\le \sum_{0}^{N-1} |u_{Q_i} - u_{Q_{i+1}}| \\
&\le \sum_{0}^{N-1} \frac{1}{m_n(Q_i)} \int_{Q_i} |u - u_{Q_{i+1}}|\, dx \\
&\le \sum_{0}^{N-1} \frac{1}{m_n(Q_i)} \int_{Q_i \cup Q_{i+1}} |u - u_{Q_{i+1}}|\, dx \\
&\le C(n,p) \sum_{0}^{N-1} \frac{1}{m_n(Q_i)} \; \frac{m_n(Q_i \cup Q_{i+1})}{m_n(Q_i)} \; \ell(Q_i) \int_{Q_i \cup Q_{i+1}} |\nabla u|\, dx \\
&= C(n,p) \int_{\bigcup_0^N Q_i} \frac{|\nabla u|}{\delta_\Omega(x)^{n-1}} dx \\
&\le C(n,p) \left(\int_\Omega |\nabla u|^p dx \right)^{\frac{1}{p}} \left(\int_{\cup Q_i} \frac{dx}{\delta_\Omega(x)^{\frac{n-1}{p-1}p}} \right)^{\frac{p-1}{p}} .
\end{aligned}
$$

Now

$$
\int\limits_{\cup Q_i} \frac{dx}{\delta_\Omega(x)^{\frac{n-1}{p-1}p}} = C(n,p) \int\limits_0^t \frac{ds}{\delta_\Omega(s,0')^{\left[\frac{n-1}{p-1}p-(n-1)\right]}} = C(n,p) \int\limits_0^t \frac{ds}{\delta_\Omega(s,0')^{\frac{n-1}{p-1}}} .
$$

Letting $L < \infty$ denote the supremum in (1), we obtain from (9) that

$$
\begin{aligned}
m_n(\Omega_t^+) &\le L k_p^\Omega\big((0,0'),(t,0')\big)^{1-p} \\
&= L \left(\int_0^t \frac{ds}{\delta_\Omega(s,0')^{\frac{n-1}{p-1}}} \right)^{1-p} \\
&\le L C(n,p) \int_\Omega |\nabla u|^p dx \,.
\end{aligned}
$$

This establishes (8) and completes the sufficiency proof of Part A. We point out that this type of "chaining argument" which we used in Case 2 is by now standard, *c.f.*, [14], [15], [23]. ∎

Remark. *The proof shows that*

$$
N_{p,\Omega}^p(B_0) \le C(n,p,M) \frac{m_n(\Omega)}{m_n(B_0)} + LC(n,p)
$$

so by Lemma 2.2,

$$
M_p^p(\Omega) \le C(n,M,p) \frac{m_n(\Omega)}{m_n(B_0)} + LC(n,p)
$$

where L denotes the supremum in (1).

PART B: If the supremum in (1) is infinite (for Ω^*) then Ω is not a p–Poincaré domain ($n-1 < p < \infty$).

Without loss of generality, we may assume that the supremum in (1) is assumed for $x > 0$, i.e., that

$$\sup_{b>0} k_p^{\Omega^*}\big((0,0'),(b,0')\big)^{p-1} m_n(\Omega_b^+) = \infty\,. \tag{10}$$

We invoke Lemma 2.4 to obtain a sequence $\langle a_i \rangle_{i\in I}$ with the properties listed therein, and with $x_0 = 0$. In particular

$$\operatorname{dist}\left[(0,0'),[\partial\Omega^*]_{a_0}\right] = \delta_{\Omega^*}(0,0')\,.$$

For $i \in I$, $i > 0$, define the set

$$A_i = \big\{x = (x_1,x') \in \mathbb{R}\times\mathbb{R}^{n-1} : x_1 \in (a_{i-1},a_i) \text{ or } x_1 = a_j \text{ and } x' \in \Omega_{x_1} \\ \text{with } j = i-1 \text{ or } i\big\}\,.$$

We also write $\kappa(x_1,x_2)$ for $k_p^{\Omega^*}\big((x_1,0'),(x_2,0')\big)$.

Lemma. *For each $i \in I$, $i > 1$ there exists a Lipschitz continuous function $f_i : A_i \longrightarrow \mathbb{R}$ having the following properties:*

(i) *$f_i(a_{i-1},x') = 0$ for all $(a_{i-1},x') \in \Omega_{a_{i-1}}$,*
(ii) *$f_i(a_i,x') = c_i$ for all $(a_i,x') \in \Omega_{a_i}$ where $C_1(n,p) \le \frac{C_i}{\kappa(a_{i-1},a_i)} \le C_2(n,p)$, and*
(iii) *$\int_{A_i} \left|\nabla f_i\right|^p dx \le C(n,p)\kappa(a_{i-1},a_i)$*

Proof of Lemma. Let $\Delta a_i = a_i - a_{i-1}$.

Case 1. $\Delta a_i \le 5\min\{\varphi(a_{i-1}),\varphi(a_i)\}$.

We perform the construction of f_i in case $\varphi(a_{i-1}) \le \varphi(a_i)$. The construction in case $\varphi(a_{i-1}) > \varphi(a_i)$ can simply be obtained by the present construction by a reflection. In this case, by Lemma 2.4 we have

$$\delta_{\Omega^*}(x,0') \approx \varphi(a_{i-1}) \text{ for } a_{i-1} \le x \le a_i$$

so that

$$\kappa(a_{i-1},a_i) = \int_{a_{i-1}}^{a_i} \frac{ds}{\delta_{\Omega^*}^{\frac{n-1}{p-1}}} \approx \Delta a_i\,\varphi(a_{i-1})^{-\frac{(n-1)}{(p-1)}}\,. \tag{11}$$

The definition of f_i will be symmetric about the axis $x_{1,i}$ which is parallel to the x_i axis and passes through the center of $\Omega_{a_{i-1}}$ meaning that on each 2–plane containing the $x_{1,i}$–axis, the definition of f_i will be the same. Fixing such a 2–plane we may now think of f_i as a function of two variables $x_{1,i}$ and y : $f_i(x_{1,i},y)$. We need an auxiliary linear function $\lambda : \mathbb{R} \longrightarrow \mathbb{R}$ which is specified by $\lambda'(t) = -1$ and $\lambda(a_i) = 0$.

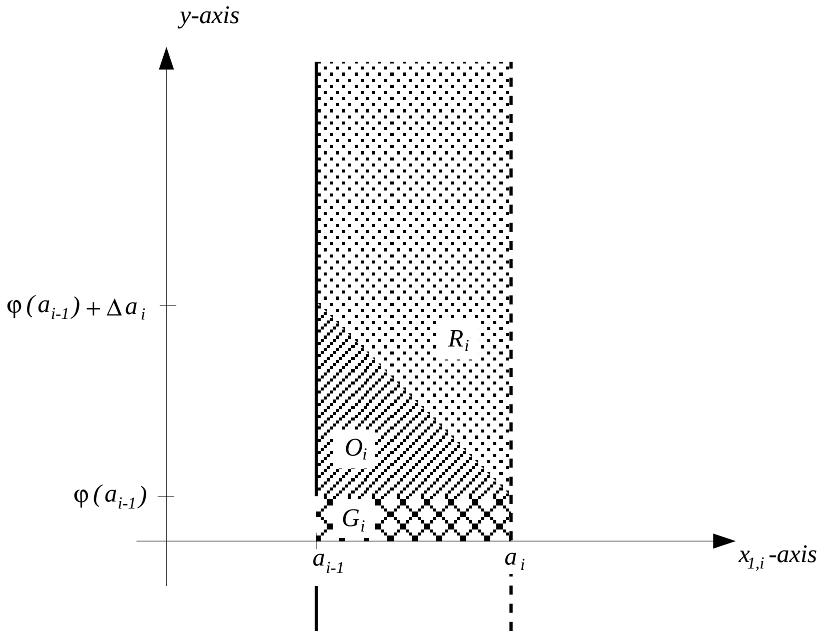

FIGURE 1. Defining the function f_i.

Referring to the regions defined in Figure 2, we define f_i as follows:

$$
\begin{aligned}
F(x_{1,i}) &:= f_i(x_{1,i}, 0) = (x_{1,i} - a_{i-1})\varphi(a_{i-1})^{-(n-1)/(p-1)} \\
\text{on } G_i &: f_i(x_{1,i}, y) = F(x_{1,i}) \\
\text{on } O_i &: f_i(x_{1,i}, y) = F(x_{1,i}) \left[1 - \frac{y - \varphi(a_{i-1})}{\lambda(x_{1,i})}\right] + F(a_i) \left[\frac{y - \varphi(a_{i-1})}{\lambda(x_{1,i})}\right] \\
\text{on } R_i &: f_i(x_{1,i}, y) = F(a_i)\,, \\
\text{and for } y < 0 &: f_i(x_{1,i}, y) = f_i(x_{1,i}, |y|)\,.
\end{aligned}
$$

Only property (iii) needs checking. To facilitate the computation we drop the subscript i from f_i, $x_{1,i}$ and further let G, O and R denote the partition of A_i obtained by revolving G_i, O_i and R_i respectively about the $x_{1,i}$–axis.

On G, $\left|\nabla f(x_1, x')\right| = F'(x_i) = \varphi(a_i)^{-(n-1)/(p-1)}$ so that (using (11))

$$
\begin{aligned}
\int_G \left|\nabla f\right|^p dx &\approx \varphi(a_i)^{n-1-p\left(\frac{n-1}{p-1}\right)} \Delta a_i \\
&= \Delta a_i \; \varphi(a_{i-1})^{-\left(\frac{n-1}{p-1}\right)} \\
&\approx \kappa(a_{i-1}, a_i)\,.
\end{aligned}
$$

On O, letting $r = |x'|$ we have,

$$
\begin{aligned}
f_{x_1}(x_1, x') &= F'(x_1) \left[1 - \frac{|x'| - \varphi(x_1)}{\lambda(x_1)}\right] + \frac{F(x_1) - F(a_i)}{\lambda(x_1)^2} \left[|x'| - \varphi(a_{i-1})\right] \\
f_r(x_1, x') &= \left[F(a_i) - F(x_1)\right] \frac{1}{\lambda(x_1)}\,.
\end{aligned}
\tag{12}
$$

Hence

$$\int_O |f_r|^p \approx \int_{a_{i-1}}^{a_i} \left| \frac{F(a_i)-F(x_1)}{\lambda(x_1)} \right|^p \left[\big(\lambda(x_i)+\varphi(a_i)\big)^{n-1} - \varphi(a_i)^{n-1} \right] dx_1 \,.$$

But since $F'(t) \approx \varphi(a_{i-1})^{-(n-1)/p-1}$ $(a_{i-1} \le t \le a_i)$ and since

$$\begin{aligned}(\lambda(x_i)+\varphi(a_i))^{n-1} - \varphi(a_i)^{n-1} &\le (n-1)(\Delta a_i + \varphi(a_{i-1}))^{n-2}\Delta a_i \\ &\lesssim \varphi(a_{i-1})^{n-1}\end{aligned}$$

we may conclude just as above that

$$\int_O |f_r|^p dx \lesssim \kappa(a_{i-1}, a_i)\,.$$

We have left to estimate $\int_O |f_{x_1}|^p$. By (12) we can write (since $\big||x'| - \varphi(a_{i-1})\big| \le \lambda(x_1)$ for $(x_1, x') \in O$)

$$\big|f_{x_1}(x_1, x')\big| \le F'(x_1) + \frac{\big|F(x_1)-F(a_i)\big|}{\lambda(x_1)}\,.$$

The $L^p(O)$-norm of the first term is comparable (since $\Delta a_i \lesssim \varphi(a_{i-1})$) to $\|F'(x_1)\|_{L^p(G)}$ which was already shown to have the desired bound. The second term is just $f_r(x_1, x')$ whose $L^p(O)$–norm was already estimated.

Case 2. $\Delta a_i > 5 \min\big\{\varphi(a_{i-1}), \varphi(a_i)\big\}$.

As in Case 1, we assume $\varphi(a_{i-1}) \le \varphi(a_i)$, and we let the $x_{1,i}$–axis be the axis pointing in the same direction as the x_1–axis and passing through the center of $\Omega_{a_{i-1}}$. Let $\widehat{a_i}$ be the ordinate on the $x_{1,i}$ axis which is equidistant to $\big[\partial\,\Omega\big]_{a_{i-1}}$ and to the $(n-1)$–plane $x_i = a_i$. Put $\widehat{\Delta} a_i = \widehat{a_i} - a_{i-1}$. For the remainder of Case 2, we shall express arguments of f_i by using the special coordinates in $\mathbb{R}^n : (x_1, x')_i$, where $x_i \in \mathbb{R}$ is the position on the $x_{1,i}$ axis and $x' \in \mathbb{R}^{n-1}$ denotes the coordinates on the other $n-1$ axes which all pass through the $x_{1,i}$ axis at $(0, 0')_i = \overrightarrow{0}_i$. We now define f_i as follows:

$$\begin{aligned}
&\text{on } A_i \cap \mathrm{Ball}^n\left(\left(a_{i-1}, 0'\right)_i, \varphi(a_{i-1})\right) : f_i \equiv 0\,,\\
&\text{on } M_i := A_i \cap \overline{\mathrm{Ball}^n\left(\left(a_{i-1}, 0'\right)_i, \widehat{\Delta} a_i\right)} \setminus \mathrm{Ball}^n\left(\left(a_{i-1}, 0'\right)_i, \varphi(a_i)\right) :\\
&\qquad\qquad f_i\big(\overrightarrow{x}_i\big) = \kappa(a_{i-1} + \varphi(a_{i-1}), r) \text{ where}\\
&\qquad r = \big|\overline{x}_i - (a_{i-1}, 0')_i\big|, \text{ and}\\
&\text{on } A_i \setminus \mathrm{Ball}^n\left(\left(a_{i-1}, 0'\right), \widehat{\Delta} a_i\right) : f_i\big(\overrightarrow{x}_i\big) = f_i\big(\widehat{a}_i, 0'\big)_i\,.
\end{aligned}$$

With this definition certainly (i) holds and since $\Delta a_i > 5\varphi(a_{i-1})$ we have $\kappa\big(a_{i-1} + \varphi(a_{i-1}), \widehat{a}_i\big) \approx \kappa(a_{i-1}, \widehat{a}_i) \approx \kappa(a_{i-1}, a_i)$ (recall $\varphi(a_{i-1}) \le \varphi(a_i)$) whence (ii) holds. To verify (iii) we compute using polar coordinates.

Observe that on M_i, (using Lemma 2.4)

$$\left|\nabla f_i\big(\overrightarrow{x}_i\big)\right| = \delta_{\Omega^*}\left(a_{i-1} + r, 0'\right)^{-(n-1)/(p-1)}$$

(the arguments of δ_{Ω^*} here and below are with respect to the standard coordinate system, as opposed to those of f_i.)

Whence, noting that for $r \in \big[\varphi(a_{i-1}), \widehat{a}_i - a_{i-1}\big]$, $r \approx \delta_{\Omega^*}\big(a_{i-1}+r, 0'\big)$, we have

$$\begin{aligned}
\int_{Ai} \left|\nabla f_i\right|^p dx &= \int_{Mi} \left|\nabla f_i\right|^p dx \\
&= c(n) \int_{\varphi(a_{i-1})}^{\widehat{a}_i - a_{i-1}} \frac{r^{n-1} dr}{\delta_{\Omega^*}\left(a_{i-1}+r, 0'\right)^{\frac{n-1}{p-1}p}} \\
&\approx c(n) \int_{\varphi(a_{i-1})}^{\widehat{a}_i - a_{i-1}} \delta_{\Omega^*}\left(a_{i-1}+r, 0'\right)^{-\frac{(n-1)}{(p-1)}} dr \\
&\leq c(n)\kappa(a_{i-1}, a_i)\,.
\end{aligned}$$

This completes the proof of the Lemma.

Consider the following restricted supremum of (10):

$$L_0 = \sup_{i>0} \kappa(0, a_i)^{p-1} m_n\left(\Omega^+_{a_i}\right). \tag{13}$$

In order to show that Ω is not a p–Poincaré domain, by Lemma 2.2, it suffices to construct functions $F \in (\mathrm{Lip}_{loc} \cap W^{1,p})(\Omega) \setminus \{0\}$ satisfying $F(x_1, x') = 0$ whenever $x_1 < 0$ and which have arbitrarily large Rayleigh–Ritz quotients:

$$R_N(F) = \frac{m_n\big(\{F=1\}\big)}{\int_\Omega \left|\nabla F\right|^p dx}.$$

Case 1. $L_0 = \infty$.

Fix $i > 1$ large enough so that $\kappa(0, a_i) < \frac{1}{2}\kappa(a_1, a_i)$. Note that by (10) it follows from Lemma 2.5 that the set $\{a_i : i > 0\}$ of Lemma 2.4 is necessarily infinite. Although this fact is an immediate consequence of (13) it is important to realize (since we will also need it in Case 2) that it does indeed follow from (10). Define $G_i : \Omega \longrightarrow \mathbb{R}$ by

$$G_i(x_1, x') = \begin{cases} f_j(x_1, x') + \sum\limits_{k=2}^{j-1} \max f_k & \text{if } a_{j-1} \leq x \leq a_j \text{ and } 1 < j \leq i, \\ 0 & \text{if } x < a_1 \\ G_i(a_i, 0') & \text{if } x > a_i\,. \end{cases}$$

Then G_i is globally Lipschitz continuous on Ω and by the Lemma,

$$\max G_i = \sum_2^i \max f_j \approx \kappa(0, a_i) \quad \text{while}$$

$$\int_\Omega \left|\nabla G_i\right|^p dx = \sum_2^i \int_{A_j} \left|\nabla f_j\right|^p dx \lesssim \kappa(0, a_i)\,.$$

Thus letting $F_i = G_i / \max G_i$, we have

$$R_N(F_i) = \frac{m_n\big(\{F_i = 1\}\big)}{\int_\Omega \left|\nabla F_i\right|^p dx} \gtrsim \frac{m_n\left(\Omega^+_{a_i}\right)}{\kappa(0, a_i)^{-p+1}}$$

so by (13), $\sup_i R_N(F_i) = \infty$.

Case 2. $L_0 < \infty$.

Let $K > 0$ be a large number. We assume in particular that $K > \tau L_0$ where $\tau > 0$ is a large positive number. Lower bounds on τ shall be specified later, as needs arise. By (10) we obtain $b > a_i$ such that

$$\kappa(0,b)^{p-1} m_n(\Omega_b^+) > K\,.$$

Choose i such that

$$a_{i-1} < b < a_i\,.$$

We also assume that i is large enough (which can be converted into a requirement on the size of τ) to insure that

$$\Delta a_i \le a_{i-1}\,. \tag{14}$$

Since by construction $\Delta a_i \le 2M$ for each i, if $\sup a_i = \infty$ certainly (14) will eventually hold. If $\sup a_i < \infty$ then since $\langle a_i \rangle_{i>0}$ is an infinite set, we must have $\Delta a_i \longrightarrow 0$ and once again it is obvious that (14) will eventually hold.

Next, since for $b > 0$

$$\begin{aligned} K < \kappa(0,b)^{p-1} m_n(\Omega_b^+) &\le c(p)\big[\kappa(0,a_{i-1})^{p-1} + \kappa(a_{i-1},b)^{p-1}\big] m_n(\Omega_b^+) \\ &\le c(p)\big[L_0 + \kappa(a_{i-1},b)^{p-1} m_n(\Omega_b^+)\big] \end{aligned}$$

we have

$$K\big(1 - c(p)\tau^{-1}\big) \le c(p)\kappa(a_{i-1},b)^{p-1} m_n(\Omega_b^+)\,. \tag{15}$$

It will now be convenient to split the remainder of the proof into two subcases as in the proof of the Lemma.

Subcase 2a. $\Delta a_i \le 5 \min\big\{\varphi(a_{i-1}), \varphi(a_i)\big\}$. In this case, by (14) (and using Lemma 2.4 to estimate κ) we have

$$\begin{aligned} \kappa(a_{i-1},b)^{p-1} m_n(\Omega_b^+) &\lesssim \kappa(0,a_{i-1})^{p-1} m_n(\Omega_b^+) \\ &\le \kappa(0,a_{i-1})^{p-1} m_n(\Omega_{a_{i-1}}^+) \\ &\le L_0 \end{aligned}$$

so (15) would yield

$$K(1 - c(p)\tau^{-1}) \le c(p) L_0 \le c(p)\tau^{-1} K\,.$$

This inequality will be contradictory as soon as τ is sufficiently large. Subcase 2a is thus dealt with.

Subcase 2b. $\Delta a_i > 5 \min\{\varphi(a_{i-1}), \varphi(a_i)\}$.

Let $\overline{a}_i = \frac{1}{2}(a_{i-1} + a_i)$.

As a first reduction, we show that we may assume $b > \overline{a}_i$ and

$$(i)\ \varphi(a_i) \le \varphi(a_{i-1}) \text{ and } (ii)\ \kappa(a_{i-1},b) \lesssim \kappa(\overline{a}_i,b)\,. \tag{16}$$

Indeed, first of all, from (14) we can conclude

$$\kappa(a_{i-1}, \overline{a}_i) \le c(n,p)\kappa(0, a_{i-1})\,. \tag{17}$$

Now, if (16) were false then we would have

$$\kappa(a_{i-1}, b) \le 2\kappa(a_{i-1}, \overline{a}_i)\,. \tag{18}$$

But we could then conclude from (15) using (17) and (18) that

$$K(1 - c(p)\tau^{-1}) \lesssim \kappa(0, a_{i-1}) m_n(\Omega^+_{a_{i-1}}) \le \tau^{-1} K$$

so once again, if τ is sufficiently large, we would have a contradiction so (16) is established.

Since

$$m_n(\Omega_b^+) \le c(n)(a_i - b)M^{n-1} + m_n(\Omega^+_{a_i})$$

we can use (15) and (16) (i) to deduce that

$$\begin{aligned} K(1 - c(p)\tau^{-1}) &\lesssim \kappa(a_{i-1}, b)^{p-1} m_n(\Omega_b^+) \\ &\lesssim \kappa(\overline{a}_i, b)^{p-1} m_n(\Omega_b^+) \\ &\lesssim \kappa(0, a_i)^{p-1} m_n(\Omega^+_{a_i}) + \kappa(\overline{a}_i, b)^{p-1}(a_i - b)M^{n-1}\,. \end{aligned} \tag{19}$$

We introduce the cylinder R_i about the x_1–axis having radius M and projection $[a_i - 2M, a_i]$ on the x_1–axis. Note that $\overline{a}_i > a_i - M$ (since $\Delta a_i \le 2M$) and consequently (by Lemma 2.4 and (16 (ii)))

$$\begin{aligned} \kappa(\overline{a}_i, b) &\lesssim \int_{\overline{a}_i}^{b} \frac{ds}{\text{dist}\,\left[(s, 0'), \left[\partial\Omega^*\right]_{a_i}\right]^{(n-1)/(p-1)}} \\ &\le k_p^{R_i}\left((\overline{a}_i, 0'), (b, 0')\right)\,. \end{aligned}$$

Combining this inequality with (19) and then applying Lemma 2.5, we obtain:

$$\begin{aligned} K\big(1 - c(p)\tau^{-1}\big) &\lesssim L_0 + k_p^{R_i}\left((a_i - M, 0'), (b, 0')\right)^{p-1} m_n\big((R_i)_b^+\big) \\ &\le \tau^{-1} K + C(M, p, n) \end{aligned}$$

which is clearly impossible as long as $K > C(M,p,n)$ and τ is sufficiently large.

PART C.

An Example. *This example will show that the range of exponents in Theorem A is best possible. We show that for each p, $1 < p \le n-1$, there exists a domain Ω in $\mathbb{R}^n$ for which the condition (1) holds but the p–Poincaré inequality fails.*

Since we may assume $n \ge 3$, we have

$$i^{-n} < \frac{1}{i} - \frac{1}{i+1}$$

for each $i \geq 2$. The Steiner symmetric domain Ω will be defined as in Theorem A by the following function $\varphi : \big[-1,1\big] \longrightarrow \big[0,1\big]$

$$\varphi(t) = \begin{cases} 1, & \text{if } \frac{1}{i} < 1-t < \frac{1}{i} + i^{-n} \text{ for } i \geq 2 \\ 1-|t|, & \text{otherwise}. \end{cases}$$

To see that (1) holds for this domain Ω and any $p \in (1, n-1]$ we first observe that $\delta_\Omega(t, 0') \approx 1 - |t| \ \big(|t| < 1\big)$ so that

$$k_p^\Omega\big((0,0'),(t,0')\big) \approx (1-|t|)^{(p-n)/(p-1)}\,.$$

On the other hand,

$$m_n\left(\Omega^+_{|t|}\right) \approx m_n\left(\Omega^-_{-|t|}\right) \approx \big(1-|t|\big)^n$$

and hence we may conclude that the supremum in (1) is dominated by $C \max\limits_{0 \leq t \leq 1} (1-t)^p \lesssim 1$.

In showing that Ω is not a p–Poincaré domain for any $p \in (1, n-1]$ we use the explicit formulas for the p–capacity of a condensor determined by a pair of concentric balls (see [Maz–85] §2.2.4). If we take the smaller $(n-1)$–dimensional ball to have radius $r < \frac{1}{2}$ and the larger one to have radius

$$R = \begin{cases} 2r, & \text{if } p < n-1 \\ \sqrt{r}, & \text{if } p = n-1\,, \end{cases}$$

these formulas give:

$$\operatorname{Cap}_p\big(\operatorname{Ball}^{n-1}(r), \operatorname{Ball}^{n-1}(R)\big) = \begin{cases} Cr^{n-1-p}, & \text{if } p < n-1 \\ C\big(\log\frac{1}{\sqrt{r}}\big)^{2-n}, & \text{if } p = n-1\,. \end{cases}$$

This means that for each $r \in (0, \frac{1}{2})$, there exists a function $U_r \in W^{1,p}\big(\operatorname{Ball}^{n-1}(1)\big) \cap \operatorname{Lip}_{\text{loc}}\big(\operatorname{Ball}^{n-1}(1)\big)$ satisfying $U_r(x') = 1$ for $|x'| > R$, $U_r(x') = 0$ for $|x'| < r$ and

$$\int_{\operatorname{Ball}^{n-1}(1)} \big|\nabla U_r(x')\big|^p d\mathcal{H}^{n-1}(x') = \begin{cases} Cr^{n-1-p} & \text{if } p < n-1 \\ C\big(\log\frac{1}{\sqrt{r}}\big)^{2-n}, & \text{if } p = n-2\,. \end{cases}$$

Aside: In fact, such a function can be explicitly written down.

Now fix $i \geq 2$, let $r_i = \frac{1}{i}$ and define a function $u_i \in W^{1,p}(\Omega) \cap \operatorname{Lip}_{\text{loc}}(\Omega)$ as follows:

$$u(x,x') = \begin{cases} U_{r_i}(x'), & \text{if } \frac{1}{i} < 1-x < \frac{1}{i} + i^{-n} \\ 0, & \text{otherwise}. \end{cases}$$

Observe $m_n\big(\{u_i = 1\}\big) \approx i^{-n}$ but $\int_\Omega \big|\nabla u_i\big|^p dx = o(i^{-n})$. Therefore by Lemma 2.2 we conclude that Ω cannot be a p–Poincaré domain.

This completes the proofs of the main results. ■

SECTION 4: AN EXTENSION TO A MORE GENERAL SOBOLEV–POINCARÉ INEQUALITY

A useful generalization of the p–Poincaré inequality is obtained by using two possibly different exponents on either side of the inequality.

Definition. *A domain $\Omega \subseteq \mathbb{R}^n$ of finite volume is said to support the (q,p)–Poincaré inequality $(1 \le p, q < \infty)$ (or Ω is called a (q,p)–Poincaré domain) if there exists a positive number C such that*

$$\|u - u_\Omega\|_{L^q(\Omega)} \le C\|\nabla u\|_{L^p(\Omega)} \tag{20}$$

holds for all functions $u \in W^{1,p}(\Omega)$.

When $p \le q$, (q,p)–Poincaré domains have a concrete characterization which is analogous to the one given in Lemma 2.2 for p–Poincaré domains.

Lemma 4.1. *Assume that $\Omega \subseteq \mathbb{R}^n$ is a domain of finite volume, let $1 \le p \le q < \infty$ and let $B_0 \subseteq \Omega$ be a ball. Then Ω is a (q,p)–Poincaré domain if and only if for each function $u \in \mathrm{Lip}_{\mathrm{loc}}(\Omega) \cap W^{1,p}(\Omega)$ which vanishes on B_0 we have:*

$$m_n\big(\{u = 1\}\big)^{p/q} \le D \int_\Omega \big|\nabla u\big|^p dx\,, \tag{21}$$

where D is a fixed positive constant.

The proof is similar to that of Lemma 2.2 (see [18] and [12]), we omit the details. In light of this lemma and Theorem A, it seems quite plausible that for a Steiner symmetric domain as considered in Theorem A and $p > n-1$, the (q,p)–Poincaré inequality might be equivalent to the following inequality:

$$\sup_{x \in \mathrm{Proj}_{x_1}(\Omega)} k_p^\Omega\big((0,0'),(x,0')\big)^{p-1} m_n\big(T(\Omega_x)\big)^{p/q} < \infty\,. \tag{22}$$

The following example shows that such a result is not possible. (One could also use a domain with an outward directed cusp of exponential order.)

Example 4.2. *Fix p, $1 < p < \infty$.*

We construct a domain $\Omega \subseteq \mathbb{R}^n$ for which (22) holds for all q, $p \le q \le q_0$, $q_0 > p$, but on which the (q,p)–Poincaré inequality fails for any $q > p$. In fact, one can take $q_0 = np/(n-p)$ when $p < n$ and q can be taken to be any finite number when $p \ge n$.

Since the domain to be constructed is Steiner symmetric, it is enough to specify a nonnegative function $\varphi(x_1)$ which gives the radius of the (ball) cross–sections Ω_{x_1}. To define $\varphi : [-1,1] \longrightarrow [0,2]$ we begin by noting that for each $i \ge 2$ we have

$$2^{-i} < \frac{1}{(i-1)} - \frac{1}{i}.$$

Next we define

$$\varphi(t) = \begin{cases} 2, & \text{if } \frac{1}{i} < t < \frac{1}{i} + 2^{-i},\ i \ge 2 \\ 1 - |t|, & \text{otherwise}\,. \end{cases}$$

To show that Ω is not a (q,p)–Poincaré domain for any $q>p$, we will use Lemma 4.1. For this we construct, for each $i\geq 2$ a function $u_i\in W^{1,p}(\Omega)\cap \mathrm{Lip}_{\mathrm{loc}}(\Omega)$ as follows:

$$u_i(x_1,x')=\begin{cases}1, & \text{if } x_1\in\left(\frac{1}{i},\frac{1}{i}+2^{-i}\right), i\geq 2 \text{ and } |x'|>\frac{5}{3}\\ 3\big(|x'|-\frac{4}{3}\big), & \text{if } x_1\in\left(\frac{1}{i},\frac{1}{i}+2^{-i}\right), i\geq 2 \text{ and } \frac{4}{3}\leq|x'|\leq\frac{5}{3}\\ 0, & \text{otherwise}\,.\end{cases}$$

Note that $\big|\nabla u_i\big|\leq c_0$, where $c_0<\infty$ is independent of i. Whence we have

$$\int_\Omega \big|\nabla u_i\big|^p dx\approx 2^{-i}$$

and

$$m_n\big(\{u_i=1\}\big)\approx 2^{-i}\,.$$

From these two relations we immediately conclude that (21) cannot hold on Ω and hence that Ω cannot be a (q,p)–Poincaré domain for any $q>p$. On the other hand, using the fact that $\delta_\Omega(x_1,0')\approx\delta_{\widetilde{\Omega}}(x_1,0')$ $\big(|x_1|<1\big)$ where $\widetilde{\Omega}$ is defined by the graph $\widetilde{\varphi}(x_1)=1-|x_1|$, a simple computation shows that (22) holds for the indicated values of q.

Despite the above example, there is a more restrictive class of Steiner symmetric domains in $\mathbb{R}^n$ for which (22) is indeed equivalent to the (q,p)–Poincaré inequality. This is the content of our next result which is a restatement of Theorem C.

Theorem 4.3. *Assume that $\Phi:\mathbb{R}\longrightarrow[0,\infty)$ is Lipschitz continuous and*

$$\Omega=\big\{(x_1,x')\in\mathbb{R}\times\mathbb{R}^{n-1}:|x'|<\Phi(x_1)\big\}=\Omega^*$$

is a domain of finite volume. We may assume $\Phi(0)>0$. Then for $p>1$ and $p\leq q\leq q_0$, Ω is a (q,p)–Poincaré domain if and only if

$$\sup_{x\in\mathrm{Proj}_{x_1}(\Omega)} k_p^\Omega\big((0,0'),(x,0')\big)^{p-1}m_n\big(T(\Omega_x)\big)^{p/q}<\infty\,. \tag{22}$$

Here q_0 can be taken to be $\frac{np}{n-p}$ when $p<n$ and q can be any positive number $(\geq p)$ when $p>n$.

Before going to the proof we record the special case of Theorem 4.3 when $q=p$.

Corollary 4.4. *If in Theorem A, the function φ is assumed to be Lipschitz continuous, then the conclusion will hold for all $p>1$.*

Proof of Theorem 4.3.

Part I: Necessity of (22) for the (q,p)–Poincaré inequality.
In uniformly bounding the quantity inside the supremum of (22), we may assume $x=t>0$. Define

$$G(x_1,x')=\begin{cases}\int_0^{x_1}\Phi(s)^{(1-n)/(p-1)}ds, & \text{if } 0\leq x_1\leq t\\ \int_0^t \Phi(s)^{(1-n)/(p-1)}ds, & \text{if } x_1>t\\ 0, & \text{if } x_1<0\end{cases}$$

and let $F = G/\max G$. Since

$$\{F=1\} = \Omega_t^+ \quad , \quad \text{and}$$
$$\int_\Omega \left|\nabla F\right|^p dx \approx \int_0^t \Phi(s)^{(1-n)/(p-1)} ds$$

we get from Lemma 4.1 the inequality

$$m_n(\Omega_t^+)^{p/q} \lesssim D\left[\int_0^t \Phi(s)^{(1-n)/(p-1)} ds\right]^{1-p}$$

which is tantamount to (22).

Part II. Sufficiency of (22) for the (q,p)–Poincaré inequality.

We consider a function u as in Lemma 2.1. Let $M = \max\Phi$. Fix $x_1 > 0$ and put $B_x = \mathrm{Ball}^n\big((x_1, 0'), \frac{1}{2}\delta_\Omega(x_1, 0')\big)$.

Case 1. $u_{x_1} := \frac{1}{m_n(B_{x_1})}\int_{B_{x_1}} u(x)dx \le \frac{1}{2}$. Letting $\Omega(x_1) = \big\{(x,x') \in \Omega : |x - x_1| \le \frac{1}{2}\delta_\Omega(x_1, 0')\big\}$, we have $\int_{\Omega(x_1)} |u - u_{x_1}|^q dx \gtrsim m_n\big(\{u=1\} \cap \Omega(x_1)\big)$. A simple calculation shows that the formula $\Psi(x,x') = \big(x, \frac{\Phi(x_1)}{\Phi(x)}\, x'\big)$ defines a bilipschitz mapping on $\Omega(x_1)$ with a fixed bilipschitz constant (independent of x_1) C_Ψ which can be taken $\lesssim L^2 + 1$, where L is the Lipschitz constant of Φ. This means that, for each pair (x,x'), $(y,y') \in \Omega(x_1)$ we have:

$$C_\Psi^{-1}\big|(x,x') - (y,y')\big| \le \big|\Psi(x,x') - \Psi(y,y')\big| \le C_\Psi\big|(x,x') - (y,y')\big|\,.$$

It is an elementary fact that bilipschitz mappings convert (q,p)–Poincaré domains to (q,p)–Poincaré domains with comparable constants (for a slightly more general result when $p = q$, we refer to [22] Lemma 7.1(d)).

Now, $\Psi(\Omega(x_1))$ is a cylinder with radius $\Phi(x_1)$ and length $\delta_\Omega(x_1, 0') \le \Phi(x_1) \le M$. Since all such cylinders have Poincaré constants which are uniformly bounded (they are bilipschitz equivalent to a ball), we may conclude that each $\Omega(x_1)$ will be a (q,p)–Poincaré domain with a constant C independent of x_1. Hence

$$\left(\int_{\Omega(x_1)} |u - u_{x_1}|^q dx\right)^{p/q} \lesssim \int_{\Omega(x_1)} |\nabla u|^p dx$$

and so

$$m_n\big(\{u=1\} \cap \Omega(x_1)\big)^{p/q} \lesssim \int_{\Omega(x_1)} \left|\nabla u\right|^p dx\,. \tag{23}$$

Case 2. $u_{x_1} > \frac{1}{2}$. Let t be the first such x_1. In this case we can use a "chaining argument" very similar to the one used in Case 2 of Part A in Section 3 to show that

$$m_n(\Omega_t^+)^{p/q} \lesssim \int_\Omega \left|\nabla u\right|^p dx \tag{24}$$

Since $p/q \le 1$ we can take a sequence $\{t_i\}_{i=1}^{\infty} \subseteq \mathrm{Proj}_{x_1}(\Omega)$ such that $\Omega = \cup \Omega(t_i)$ and $\Sigma \chi_{\Omega(t_i)} \le K < \infty$ on Ω and sum up the corresponding inequalities (23) and (24) to obtain

$$m_n\big(\{u=1\}\big)^{p/q} \lesssim \int_\Omega \left|\nabla u\right|^p dx$$

and the result is established. ■

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