PLANAR POINCARÉ DOMAINS: GEOMETRY AND STEINER SYMMETRIZATION

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Abstract. We determine geometric necessary and sufficient conditions on a class of strip-like planar domains in order for them to satisfy the Poincaré inequality with exponent p, where $1 \le p < \infty$. The characterization uses hyperbolic geodesics in the domain and a metric which depends on p and generalizes the quasi-hyperbolic metric in the case $p = 2$. As an application, we show that the Poincaré inequality is preserved under Steiner symmetrization of these domains but not in general.

We also show that the our geometric condition is preserved under bounded length distortion (BLD) mappings of a domain and thus extend the class of domains for which our characterization is valid.

1. Introduction

For a $b > 0$, we call a connected planar domain Ω a b-strip provided that for each real x the cross-section $\Omega_x = \{y : (x, y) \in \Omega\}$ is either the empty set or else an interval of length no greater than b. Obviously, a b-strip is simply connected since every boundary point can be connected to infinity by a vertical ray.

Given a planar domain Ω with finite area, $m_2(\Omega) < \infty$, we say that Ω is a p-Poincaré domain provided that

$$\sup_u \frac{\int\limits_\Omega |u - u_\Omega|^p \, dm_2}{\int\limits_\Omega |\nabla u|^p \, dm_2} = M_p^p(\Omega) < \infty$$

holds, where the supremum is taken over all nonconstant functions u in the Sobolev space $W^{1,p}(\Omega)$ and u_Ω denotes the m_2-average value of u over Ω. Meyers and Serrin [MySer] have shown that $C^1(\Omega) \cap W^{1,p}(\Omega)$ is dense in $W^{1,p}(\Omega)$, so one only needs to consider such functions to establish that a domain Ω is a p-Poincaré domain.

We define a metric on Ω for each $1 < p < \infty$. The metric $k_{p,\Omega}$ on Ω is defined by

$$k_{p,\Omega}(z_1, z_2) = \inf_\gamma \int\limits_\gamma \frac{ds}{\delta_\Omega^{1/(p-1)}} \,,$$

1991 *Mathematics Subject Classification.* Primary 46E35 Secondary 30C60, 30E10, 41A99.

Key words and phrases. Sobolev spaces, Hyperbolic metric, Poincaré Inequality, symmetrization.

The first author is supported in part by a grant from the National Science Foundation.

where the infimum is taken over all rectifiable arcs γ joining z_1 to z_2 in Ω, $\delta_\Omega(z)$ denotes the Euclidean distance from z to the boundary of Ω and ds denotes integration with respect to arc length. The metric will be denoted simply by k_p when the domain of definition is clear. These metrics were introduced in [SmSt90], where they were used to investigate the Poincaré inequality on domains in $\mathbb{R}^n$.

Assume from now on that Ω is simply connected domain with nonempty boundary. Then, Ω is conformally equivalent to the unit disk and hence we can define the hyperbolic metric on Ω by means of this equivalence and the geodesics in this metric are just the images of the real axis under the class of Riemann mappings onto Ω. Observe that when $p = 2$ we have that k_2 is the well-known quasi-hyperbolic metric introduced by Gehring and Palka in [GePa]. It is a consequence of the Koebe distortion theorems that these two metrics are comparable.

Fix a simply connected domain Ω with finite area and let $w_0 \in \Omega$. An open Jordan arc C in Ω is a *separating arc* provided $\Omega \setminus C$ has two components. If $w_0 \notin C$, then we denote by $\Omega(C)$ the component not containing w_0. Of course, every crosscut of Ω is a separating arc but the converse is false since the ends of a separating arc may not converge to a pair of points in $\partial\Omega$. Every hyperbolic geodesic is a separating arc and we will be interested in the following geometric quantity on Ω:

$$K_{p,\Omega}(w_0) = \sup_{\tau} k_p^{p-1}(w_0, \tau) \cdot m_2(\Omega(\tau))$$

where the supremum is taken over all hyperbolic geodesics τ with $w_0 \notin \tau$. For a b-strip Ω we have that Ω_x is a vertical crosscut whenever it is nonempty. Put $x_0 = \Re w_0$, then we define

$$V_{p,\Omega}(w_0) = \sup_{L} k_p^{p-1}(w_0, L) \cdot m_2(\Omega(L))$$

where the supremum is taken over all vertical crosscuts Ω_x with $x \neq x_0$. In the case $p = 1$, the k_1-metric is not defined; nevertheless, in §6 we show that there are natural analogs to the above quantities.

With these extended definitions we now state our main results:

Theorem A. *Let Ω be a b-strip with finite area and $1 \leq p < \infty$. The following are equivalent:*

(a) *Ω is a p-Poincaré domain.*
(b) *$K_{p,\Omega}(w_0) < \infty$.*
(c) *$V_{p,\Omega}(w_0) < \infty$.*

Theorem B. *Let Ω be any simply connected domain with finite area and $1 \leq p < \infty$. A necessary condition for Ω to be a p-Poincaré domain is that $K_{p,\Omega}(w_0)$ be finite for every $w_0 \in \Omega$.*

Finally, if Ω is a domain for which $m_1(\Omega_x) < \infty$ for all real x then we define its *Steiner symmetrization* to be the domain

$$\Omega^\star = \left\{ x + iy : |y| < \frac{1}{2} m_1(\Omega_x) \text{ and } \Omega_x \neq \emptyset \right\}.$$

It is natural to ask if Steiner symmetrization preserves p-Poincaré domains. Pólya [Pól] showed this to be true for an analogous problem in which functions are normalized by the assumption that they vanish on the boundary of the domain. Surprisingly, in our case where the functions have mean value zero, the answer is no! We give examples of this in §5 and §6. Nevertheless, we do have the following results:

Theorem C. *Let Ω be a planar domain with finite area.*

(a) *Suppose that Ω be a b-strip and that $1 < p < \infty$. If Ω is a p-Poincaré domain, then so is $\Omega^\star$.*
(b) *Suppose that $m_1(\Omega_x) \le b < \infty$ for all x. If Ω is a 1-Poincaré domain, then so is $\Omega^\star$.*

These theorems are motivated by and considerably generalize several of the results in [EvHar], [SmSt87] and [SmSt90]. In particular, in [SmSt87] a version of Theorem A, for $p = 2$, was given for certain types of domains and a counterexample was constructed showing that some geometric restrictions on the domain are necessary for the equivalence of (a) and (b). See also the end of §7 where comparisons are made between our results and these earlier works.

We also point out that Maz'ja has given a capacitary characterization of p-Poincaré domains; see Theorem 1 of §4.4.3 in [Maz85]. Unfortunately, it is not clear how to translate Maz'ja's condition into a geometric condition on Ω. In §4 we explicitly construct a function which does determine the capacity of one of Maz'ja's condensers.

The relationship between the quantities $V_{p,\Omega}(w_0)$ and $K_{p,\Omega}(w_0)$, where Ω is a b-strip, is developed in §2, and the sufficiency of $V_{p,\Omega}(w_0) < \infty$ for the p-Poincaré inequality to hold on such a domain Ω is established in §3. We remark that the proof of sufficiency could alternately have been based on methods developed in [SmSt87]. The proof of the necessity of $K_{p,\Omega}(w_0) < \infty$ for a simply connected domain to be p-Poincaré domain is in §4, and our results on the effect symmetrization has on the p-Poincaré inequality are in §5. Throughout these sections it is assumed that $p > 1$. The Poincaré inequality with exponent $p = 1$ requires different techniques, and §6 is devoted to this case. Finally, the paper concludes with a section in which we extend our results on b-strips to domains that are bilipschitzian images of them.

2. Comparing Vertical Crosscuts with Hyperbolic Geodesics

For a simply connected domain Ω with nonempty boundary we define the hyperbolic metric by

$$\rho_\Omega(w_1, w_2) = \rho_{\mathbb{D}}(z_1, z_2)$$

where the z_i's corresponds to the w_i's under a conformal mapping of the unit disk $\mathbb{D}$ onto Ω and

$$\rho_{\mathbb{D}}(z_1, z_2) = \inf_{\gamma} \int_{\gamma} \frac{2|dz|}{1 - |z|^2},$$

where the infimum is taken over all rectifiable arcs connecting z_1 to z_2 in $\mathbb{D}$. This metric is conformally invariant and the geodesics are the image of $(-1, 1)$ under arbitrary Möbius transformation of $\mathbb{D}$. Since $\rho_{\mathbb{D}}$ is conformally invariant so is ρ_Ω

and hence it does not depend on the conformal map in its definition, see [Ahl] or Chapter 4.6 of [Pom92].

From the theory of conformal mapping, we will need the following four results. The first is a conformally invariant version of Theorem 10.8 on page 311 in [Pom75]. The second is a consequence of the distortion theorem, see Chapter 2 [Pom75]. The third result provides a useful geometric property of hyperbolic geodesics. The fourth result is a theorem of Gehring and Hayman, see [GeHa].

Throughout the remainder of the paper, we use the symbols "$\approx$", "$\lesssim$" and "$\gtrsim$" to mean, respectively, "equal to", "less than or equal to" and "greater than or equal to", "modulo a multiplicative constant which depends (at worst) on the parameter p", e.g., for nonnegative a, b we have $(a+b)^p \lesssim a^p + b^p$. We denote the arc length of a rectifiable curve γ by $\Lambda(\gamma)$. More generally, we will denote by $\Lambda(E)$ the Hausdorff 1-dimensional measure of a set E.

Lemma 2.1. *Let $f : \mathbb{D} \to \Omega$ be a conformal mapping. If $a \in \mathbb{D}$, and $\epsilon > 0$ is given, then there exists a set $E = E(f, \epsilon, a) \subset \partial\mathbb{D}$ of harmonic measure $\omega_a(E) < \epsilon$; with respect to a, with the following property: If $\gamma(a, e^{i\theta})$ denotes the hyperbolic ray in $\mathbb{D}$ from a to $e^{i\theta}$ then for each $e^{i\theta} \in \partial\mathbb{D} \setminus E$, we have*

$$\Lambda(f(\gamma(a, e^{i\theta}))) < c(\epsilon)\delta_\Omega(f(a)),$$

where, as indicated, the constant $c(\epsilon)$ depends only on ϵ.

Lemma 2.2. *Let Ω be a simply connected domain with nonempty boundary. Then,*

(a) $\frac{1}{2}\rho_\Omega(w_1, w_2) \le k_2(w_1, w_2) \le 2\rho_\Omega(w_1, w_2)$ *for all $w_1, w_2 \in \Omega$, and*
(b) *if w_1 and w_2 are points Ω and with $\rho_\Omega(w_1, w_2) = 1/4$ and if γ is the hyperbolic geodesic arc in Ω with endpoints w_1 and w_2, then*

$$\Lambda(\gamma) \approx \delta_\Omega(w_1) \quad \text{and} \quad \delta_\Omega(w) \approx \delta_\Omega(w_1)$$

for all $w \in \gamma$.

Lemma 2.3. *Let Ω be a simply connected domain with nonempty boundary. Suppose that γ is a hyperbolic geodesic in Ω containing the points w_-, w_0, and w_+ where the hyperbolic distance between w_0 and either of these points is 1/4. Let τ_- (τ_+) denote the (uniquely determined) hyperbolic geodesic through w_- (w_+) which is orthogonal to γ. Then, there is a hyperbolic geodesic crosscut τ through w_0 which is disjoint from τ_- and τ_+ and satisfies:*

$$\Lambda(\tau) \approx \delta_\Omega(w_0).$$

Remark. The angle τ makes with γ at the point w_0 is close to $\pi/2$ and we will refer to this geodesic as being nearly orthogonal to γ.

Proof. Let $f : \mathbb{D} \to \Omega$ be a conformal mapping of the disk onto Ω which maps zero to w_0, $(-1, 1)$ onto γ and a point $0 < r_+ < 1$ onto w_+. Then, $f(-r_+) = w_-$ and $f^{-1}(\tau_-)$, $f^{-1}(\tau_+)$ are hyperbolic geodesics in $\mathbb{D}$. Hence, they are circular arcs which are orthogonal to both $\partial\mathbb{D}$ and the real axis. Since

$$\log \frac{1 + r_+}{1 - r_+} = \rho_\mathbb{D}(0, r_+) = \rho_\Omega(w_0, w_+) = \frac{1}{4}$$

we see that these hyperbolic geodesics in $\mathbb{D}$ divide $\partial\mathbb{D}$ into four subarcs of approximately the same arc length. Hence the harmonic measure of each of these arcs, evaluated at the origin, is bounded from zero by a positive absolute constant.

Applying Lemma 2.1 we easily produce a hyperbolic geodesic in Ω with the required length by choosing an appropriate diameter of $\mathbb{D}$. Finally, it is clear that any separating curve of Ω which has finite length must also be a crosscut. $\Box$

Lemma 2.4(Gehring-Hayman Theorem). *Let Ω be a simply connected domain with nonempty boundary. Given an arc α (with distinct endpoints) in Ω, we let γ denote the hyperbolic geodesic arc of Ω with the same endpoints as α. Then*

$$\Lambda(\gamma) \lesssim \Lambda(\alpha)\,.$$

Let Ω be a simply connected planar domain with nonempty boundary. Suppose that γ is a hyperbolic geodesic containing the distinct points w_0, w. Let τ_0 (τ) denote a hyperbolic geodesic through w_0 (w) which is nearly orthogonal to γ and has length comparable to $\delta_\Omega(w_0)$ $(\delta_\Omega(w))$ as in Lemma 2.3. Let $\gamma(w_0, w)$ be the part of γ connecting these points in Ω. We then have the following two lemmas:

Lemma 2.5. *Suppose that $\rho_\Omega(w_0, w) = 1$ and that σ is any arc in Ω with one endpoint in τ_0 and the other in τ. Then*

$$\Lambda(\sigma) \gtrsim \Lambda(\gamma(w_0, w)) \approx \delta_\Omega(w_0).$$

Proof. Let w_1 be the hyperbolic midpoint between w_0 and w. By Lemma 2.3 the hyperbolic geodesic τ_- which is orthogonal to γ at the hyperbolic midpoint between w_0 and w_1 satisfies $\tau_0 \cap \tau_- = \emptyset$. Similarly, the hyperbolic geodesic τ_+ orthogonal to γ at the hyperbolic midpoint between W_1 and w satisfies $\tau_+ \cap \tau = \emptyset$. Let σ' be any hyperbolic geodesic arc with one endpoint in $\sigma \cap \tau_-$ and the other in $\sigma \cap \tau_+$.

Now let $f : \mathbb{D} \to \Omega$ be a conformal mapping of $\mathbb{D}$ onto Ω which maps the origin to w_1 and $(-1, 1)$ onto γ. The geodesics τ_+, τ_- correspond to circular subarcs of circles which are orthogonal to $\partial\mathbb{D}$ and the real axis. Similarly, σ' corresponds to circular subarc of a circle orthogonal to $\partial\mathbb{D}$ which intersects the other two. Simple geometry and the conformal invariance of the hyperbolic metric show that there is a point $w_1' \in \sigma'$ with $\rho_\Omega(w_1, w_1') \approx 1$.

Clearly, the endpoints of σ' are a hyperbolic distance at least $1/2$ apart and hence $\Lambda(\sigma') \gtrsim \delta_\Omega(w_1')$ by Lemma 2.2(b). Now the Gehring-Hayman theorem and Lemma 2.2(b) again shows that

$$\Lambda(\sigma) \gtrsim \Lambda(\sigma') \gtrsim \delta_\Omega(w_1') \approx \delta_\Omega(w_0) \approx \Lambda(\gamma(w_0, w))$$

which proves the lemma. $\Box$

The metrics k_p depend, of course, on p. Nevertheless, the best curves to be used in the computation of this distance are essentially the same as the hyperbolic geodesics as is shown by the following lemma.

Lemma 2.6. *If $\rho_\Omega(w_0, w) \geq 1$, then for each p, $1 < p < \infty$, we have*

$$\int_{\gamma(w_0,w)} \frac{ds}{\delta_\Omega^{1/(p-1)}} \approx k_p(w_0, w) \approx k_p(w_0, \tau) \approx k_p(\tau_0, \tau). \tag{1}$$

Moreover, the first relation is true even when $\rho_\Omega(w_0, w) < 1$.

Proof. Assume that $\rho_\Omega(w_0, w) \geq 1$. Let n be the positive integer determined by $n \leq \rho_\Omega(w_0, w) < n + 1$. Denote by $w_1, \ldots, w_n$ the points on $\gamma(w_0, w)$ determined by the relation $\rho_\Omega(w_0, w_m) = m$ and $\delta_m = \delta_\Omega(w_m)$. Let γ_m be the portion of γ between w_{m-1} and w_m for $1 \leq m \leq n$. By Lemma 2.2,

$$\Lambda(\gamma_m) \approx \delta_m \quad \text{and} \quad \delta_\Omega(w) \approx \delta_m \tag{2}$$

for all $w \in \gamma_m$.

By Lemma 2.3 there is a hyperbolic geodesic τ_m which contains the point w_m, whose arc length is comparable to δ_m and is nearly orthogonal to γ, for $1 \leq m \leq n$.

Let $w_0' \in \tau_0$, $w' \in \tau$, and γ' be any rectifiable curve from w_0' to w'. We divide γ' into n subarcs where γ_m' denotes the subarc that starts at a point on τ_{m-1} and ends at a point on τ_m. By our construction, there must exist a point $w_m' \in \gamma_m' \cap \tau_m$ and hence where $\delta_\Omega(\widetilde{w}_m) \lesssim \delta_m$ since the length of τ_m is comparable to δ_m.

By Lemma 2.5, $\delta_m \lesssim \Lambda(\gamma_m')$ for all $1 \leq m \leq n$. Combining this fact with the above, we see that there is a subarc of γ_m' of length comparable to δ_m on which $\delta_\Omega(z) \lesssim \delta_m$. Hence, by the above and (2) we have

$$\begin{aligned}\int_{\gamma'} \frac{ds}{\delta_\Omega^{1/(p-1)}} &\geq \sum_{m=1}^{n} \int_{\gamma_m'} \frac{ds}{\delta_\Omega^{1/(p-1)}} \gtrsim \sum_{m=1}^{n} \frac{\delta_m}{\delta_m^{1/(p-1)}} \\ &\gtrsim \sum_{m=1}^{n} \int_{\gamma_m} \frac{ds}{\delta_\Omega^{1/(p-1)}} = \int_{\gamma(w_0,w)} \frac{ds}{\delta_\Omega^{1/(p-1)}}.\end{aligned}$$

Taking infimums we obtain that $k_p(\tau_0, \tau)$ dominates the left-hand side of (1). Thus,

$$k_p(\tau_0, \tau) \leq k_p(w_0, \tau) \leq k_p(w_0, w) \leq \int_{\gamma(w_0,w)} \frac{ds}{\delta_\Omega^{1/(p-1)}} \lesssim k_p(\tau_0, \tau)$$

and (1) follows.

Finally, suppose now that $\rho_\Omega(w_0, w) < 1$. Let σ be any curve in Ω with endpoints w_0 and w. By the Gehring-Hayman theorem, $\Lambda(\sigma) \gtrsim \Lambda(\gamma(w_0, w))$, and hence by Lemma 2.2 we have

$$\int_{\sigma} \frac{ds}{\delta_\Omega^{1/(p-1)}} \gtrsim \frac{\Lambda(\gamma(w_0, w))}{\delta_0^{1/(p-1)}} \approx \int_{\gamma(w_0,w)} \frac{ds}{\delta_\Omega^{1/(p-1)}} \geq k_p(w_0, w).$$

Taking infimums over all such σ gives comparability of the above quantities which proves the lemma. $\square$

Now let Ω be a b-strip. Since Ω is connected, the set $\{x : \Omega_x \neq \emptyset\}$ must be an interval (α, β), where $-\infty \leq \alpha, \beta \leq \infty$. Let $\alpha_n \searrow \alpha$ and $\beta_n \nearrow \beta$. Thus, we have two nested sets of crosscuts; namely, $\{\Omega_{\alpha_n}\}$ and $\{\Omega_{\beta_n}\}$. It follows that there exists a hyperbolic geodesic γ in Ω, parametrized by $(-1, 1)$, with the property that

$$\lim_{t \searrow -1} \Re\gamma(t) = \alpha \quad \text{and} \quad \lim_{t \nearrow 1} \Re\gamma(t) = \beta.$$

We will call any hyperbolic geodesic γ satisfying the above a *centerline* for the b-strip Ω.

Theorem 2.7. *Suppose that Ω is a b-strip, γ is a centerline for Ω and $w_0 \in \gamma$. Then, for $1 < p < \infty$*

$$V_{p,\Omega}(w_0) \lesssim K_{p,\Omega}(w_0) + b^p.$$

Proof. We let L be a vertical crosscut with abscissa $x \neq \Re w_0$. Without loss of generality we will assume that $x > \Re w_0$. Let w_2 be the point where γ first intersects L (starting at w_0). By Lemma 2.6

$$k_p(w_0, w_2) \approx \int_{\gamma(w_0,w_2)} \frac{ds}{\delta_\Omega^{1/(p-1)}}.$$

Next, we single out the point $w_1 \in \gamma$ which satisfies

$$\int_{\gamma(w_0,w_1)} \frac{ds}{\delta_\Omega^{1/(p-1)}} = \int_{\gamma(w_1,w_2)} \frac{ds}{\delta_\Omega^{1/(p-1)}}.$$

First assume that the hyperbolic distance from w_0 to w_1 is at least 1. By Lemma 2.3 and Lemma 2.6, we let τ_1 denote a hyperbolic geodesic which is nearly orthogonal to γ, passes through w_1 and satisfies

$$\Lambda(\tau_1) \approx \delta_\Omega(w_1) \quad \text{and} \quad k_p(w_0, \tau_1) \approx k_p(w_0, w_1). \tag{3}$$

Case 1. $\tau_1 \cap L = \emptyset$.

In this case, we find from (3), Lemma 2.6, and the choice of w_1 that

$$k_p(w_0, L) \leq k_p(w_0, w_2) \approx \int_{\gamma(w_0,w_2)} \frac{ds}{\delta_\Omega^{1/(p-1)}} = 2\int_{\gamma(w_0,w_1)} \frac{ds}{\delta_\Omega^{1/(p-1)}} \approx k_p(w_0, \tau).$$

By our construction, $w_1 \notin \Omega(L)$ so we clearly have that $\Omega(L) \subset \Omega(\tau_1)$. Hence, we obtain

$$k_p^{p-1}(w_0, L) \cdot m_2(\Omega(L)) \lesssim k_p^{p-1}(w_0, \tau_1) \cdot m_2(\Omega(\tau_1)),$$

which is the desired inequality without the extra b^p term.

Case 2. $\tau_1 \cap L \neq \emptyset$.

First we choose a point $w^\star \in \gamma(w_1, w_2)$ which satisfies

$$\delta_\Omega(w^\star) = \min\{\delta_\Omega(w) : w \in \gamma(w_1, w_2)\}.$$

Since $\Lambda(\tau_1) \approx \delta_\Omega(w_1) < b$ and $\Lambda(L) \leq b$, it follows that there is an arc in Ω of length $\lesssim b$ which connects w_1 to w_2. Thus, the Gehring-Hayman theorem implies that

$$\Lambda(\gamma(w_1, w_2)) \lesssim b.$$

The preceding inequality, in conjunction with Lemma 2.6, and the way in which we chose w_1 leads to

$$k_p(w_0, w^\star) \approx \int\limits_{\gamma(w_0,w^\star)} \frac{ds}{\delta_\Omega^{1/(p-1)}} \approx \int\limits_{\gamma(w_1,w_2)} \frac{ds}{\delta_\Omega^{1/(p-1)}} \lesssim \frac{b}{\delta_\Omega(w^\star)^{1/(p-1)}}. \tag{4}$$

The hyperbolic geodesic which we shall use for comparison will be a geodesic $\tau^\star$ which is nearly orthogonal to γ at the point $w^\star$ and satisfies $\Lambda(\tau^\star) \approx \delta_\Omega(w^\star)$.

Let $f : \mathbb{D} \to \Omega$ be a conformal map which maps $(-1, 1)$ onto γ with $\Re f(t) \to \beta$ as $t \to 1$. Since $\tau^\star$ is a crosscut, it has endpoints $\zeta^+, \zeta^- \in \partial\Omega$; where ζ^+ corresponds (under the map f) to a point of $\partial\mathbb{D}$ with positive imaginary part. A straightforward application of the Jordan curve theorem shows that the rays:

$$Z^+ = \{w : \Re w = \Re\zeta^+, \Im w \geq \Im\zeta^+\} \text{ and } Z^- = \{w : \Re w = \Re\zeta^-, \Im w \leq \Im\zeta^-\}$$

are contained in the complement of Ω. As a consequence we have the following: if $w \in \Omega$ and $\Re w \geq \max\{\Re\zeta : \zeta \in \tau^\star\}$, then $w \in \Omega(\tau^\star)$.

If $\tau^\star \cap L$ is nonempty then by the above we see that $\Omega(L) \setminus \Omega(\tau^\star)$ is contained in a vertical strip of width comparable to $\delta_\Omega(w^\star)$. On the other hand, if $\tau^\star$ is disjoint from L, then $\Omega(L) \subset \Omega(\tau^\star)$ since $w^\star \notin \Omega(L)$. In either case we have that

$$m_2(\Omega(L)) \lesssim m_2(\Omega(\tau^\star)) + b\delta_\Omega(w^\star)\,.$$

From (4), the definition of w_2 and Lemma 2.6 we have,

$$\begin{aligned} k_p^{p-1}(w_0, L) \cdot m_2(\Omega(L)) &\lesssim k_p^{p-1}(w_0, w^\star)(m_2(\Omega(\tau^\star)) + b\delta_\Omega(w^\star)) \\ &\lesssim k_p^{p-1}(w_0, \tau^\star) \cdot m_2(\Omega(\tau^\star)) + \frac{b^{p-1}}{\delta_\Omega(w^\star)} b\delta_\Omega(w^\star) \leq K_{p,\Omega}(w_0) + b^p. \end{aligned}$$

Finally, suppose that the hyperbolic distance between w_0 and w_1 is less than 1. Then, there is a point $w_3 \in \gamma$ beyond w_1 for which $\rho_\Omega(w_0, w_3) = 1$. Let τ_3 be a hyperbolic geodesic which is nearly orthogonal to γ at w_3 and satisfies $\Lambda(\tau_3) \approx \delta_\Omega(w_3)$. Put $\delta_0 = \delta_\Omega(w_0)$. By Lemma 2.2(b) and Lemma 2.6 it follows that

$$\frac{\delta_0}{\delta_0^{1/(p-1)}} \approx \int\limits_{\gamma(w_0,w_3)} \frac{ds}{\delta_\Omega^{1/(p-1)}} \approx k_p(w_0, w_3) \approx k_p(w_0, \tau_3)$$

and that $\Omega(L) \setminus \Omega(\tau_3)$ is contained in a vertical strip of width comparable to δ_0. Since $k_p(w_0, L) \approx k_p(w_0, w_1)$, we have

$$\begin{aligned} k_p(w_0, L)^{p-1} \cdot m_2(\Omega(L)) &\lesssim k_p(w_0, \tau_3)^{p-1}(m_2(\Omega(\tau_3)) + b\delta_0) \\ &\lesssim k_p(w_0, \tau_3)^{p-1} m_2(\Omega(\tau_3)) + b\delta_0^{p-1} \\ &\lesssim K_{p,\Omega}(w_0) + b^p\,. \end{aligned}$$

This completes the proof. $\square$

3. b-Strips and the Poincaré Inequality

In this section we prove that (b) implies (a) of Theorem A in the introduction. Our technique will be to decompose Ω into a countable collection W of dyadic Whitney squares (with disjoint interiors) on which the Poincaré inequality clearly holds and then use the geometry of Ω to combine these estimates. More precisely, we assume that any domain Ω is the disjoint union of squares $Q \in W$ of the form

$$Q = \{x + iy : a \le x < a + 2^{-n} \text{ and } b \le y < b + 2^{-n}\}$$

where $2^n a$, $2^n b$ and n are integers. We denote by $d(Q)$ the side length 2^{-n} of Q and let $w_Q = a + ib$. Moreover, neighboring squares are of comparable size and the quantities $d(Q)$ and $\delta_\Omega(w_Q)$ are comparable. See Chapter 6 of Stein's book [Ste] for the existence of such a decomposition.

We will use a different normalization for the Poincaré inequality in place of $M_p(\Omega)$. Fix a domain Ω of finite area and a point $w_0 \in \Omega$. Denote by D_0 the disk $D(w_0, r_0) \subset \Omega$ where $r_0 \le \frac{1}{2}\delta_\Omega(w_0)$. Put

$$N_{p,\Omega}^p(D_0) = \sup_u \frac{\int\limits_\Omega |u|^p \, dm_2}{\int\limits_\Omega |\nabla u|^p \, dm_2}$$

where the supremum is over all $u \in C^1(\Omega) \cap W^{1,p}(\Omega)$ that vanish on D_0. The following two results are Lemma 5 and Lemma 8 in [SmSt90].

Lemma 3.1. *Let Ω be as above and $1 \le p < \infty$. Then,*

$$M_p(\Omega) \lesssim N_{p,\Omega}(D_0) \lesssim \Big(\frac{m_2(\Omega)}{m_2(D_0)}\Big)^{1/p} M_p(\Omega).$$

Lemma 3.2. *Let Ω and W be as above. If $Q_1, \dots Q_n$ is a chain of squares in W, i.e., the side of Q_{j-1} is contained in the side of Q_j or vice-versa, then*

$$|u_{Q_n} - u_{Q_1}| \lesssim \sum_{j=1}^n \frac{1}{d(Q_j)} \int\limits_{Q_j} |\nabla u|.$$

Lemma 3.3. *Let Ω be a b-strip containing the square $Q = (0,a) \times (0,a)$ and $1 \le p < \infty$. If $\Omega_x = \emptyset$ for all $x \notin (0,a)$, then*

$$\int\limits_\Omega |u - u_\Omega|^p \lesssim \int\limits_\Omega |u - u_Q|^p \lesssim b^p \int\limits_\Omega |\nabla u|^p$$

whenever $u \in W^{1,p}(\Omega)$.

Proof. Let $u \in C^1(\Omega) \cap W^{1,p}(\Omega)$ then by a familiar argument we have for $1 \le p < \infty$ that

$$\int\limits_\Omega |u - u_\Omega|^p \lesssim \int\limits_\Omega |u - u_Q|^p \, dm_2 + |u_\Omega - u_Q|^p m_2(\Omega) \le 2 \int\limits_\Omega |u - u_Q|^p.$$

Now consider $w_1 = (x_1, y_1) \in \Omega$ and $w_2 = (x_2, y_2) \in Q$. By Calculus we have

$$\begin{aligned}|u(x_1,y_1) - u(x_2,y_2)|^p &\lesssim |u(x_1,y_1) - u(x_1,y_2)|^p + |u(x_1,y_2) - u(x_2,y_2)|^p \\ &\le b^{p-1} \int\limits_{\Omega_{x_1}} |\nabla u(x_1,t)|^p \, dt + a^{p-1} \int_0^a |\nabla u(s,y_2)|^p \, ds.\end{aligned}$$

Hence

$$\int\limits_{\Omega} |u - u_Q|^p \, dm_2(w_2) \le \int\limits_{Q} \frac{dm_2(w_2)}{a^2} \int\limits_{\Omega} |u(w_1) - u(w_2)|^p \, dm_2(w_1)$$

$$\lesssim \int\limits_{Q} \frac{dm_2(w_2)}{a^2} \int\limits_{\Omega} \left(b^{p-1} \int\limits_{\Omega_{x_1}} |\nabla u(x_1,t)|^p \, dt + a^{p-1} \int_{-a}^{a} |\nabla u(s,y_2)|^p \, ds \right) dm_2(w_1)$$

$$= \int\limits_{\Omega} |\nabla u(w_1)|^p m_1(\Omega_{x_1}) b^{p-1} \, dm_2(w_1) + a^{p-2} m_2(\Omega) \int\limits_{Q} |\nabla u|^p \, dm_2$$

$$\lesssim b^p \int\limits_{\Omega} |\nabla u|^p \, dm_2.$$

The result now follows by combining the above two inequalities and taking limits. $\square$

Remark. Observe that we have proved that $M_p(\Omega) \lesssim b$ for the regions described in the lemma. This contains the classical fact that $M_p(Q) \lesssim d(Q)$ whenever Q is a square.

Lemma 3.4. *Let $a_m \ge 0$, $b_m \ge 0$ and $c_m > 0$ for all $m = 0, 1, \dots$ and $1 < p < \infty$. Put $A_m = \sum_0^m a_j$, $B_m = \sum_m^\infty b_j$ and $C_m = \sum_0^m c_j$ then*

$$\sum_{m=0}^{\infty} A_m^p b_m \lesssim [\sup_m C_m^{p-1} B_m] \sum_{m=0}^{\infty} \frac{a_m^p}{c_m^{p-1}}.$$

Proof. Let $A_{-1} = C_{-1} = 0$, $q = p/(p-1)$ and $K = \sup C_m^{p-1} B_m$. For simplicity we assume that only finitely many a_j's are nonzero. By summation by parts, the mean value theorem and Hölder's inequality we get

$$\begin{aligned}\sum_{m=0}^{\infty} A_m^p b_m &= \sum_{m=0}^{\infty} A_m^p (B_m - B_{m+1}) = \sum_{m=0}^{\infty} (A_m^p - A_{m-1}^p) B_m \\ &\le p \sum_{m=0}^{\infty} A_m^{p-1} a_m B_m \qquad (1) \\ &\le p \big(\sum_{m=0}^{\infty} \frac{a_m^p}{c_m^{p-1}} \big)^{1/p} \big(\sum_{m=0}^{\infty} c_m A_m^p B_m^q \big)^{1/q}.\end{aligned}$$

Moreover,

$$\begin{aligned}
\sum_{m=0}^{\infty} c_m A_m^p B_m^q &= \sum_{m=0}^{\infty} (C_m - C_{m-1}) A_m^p B_m^q = \sum_{m=0}^{\infty} C_m (A_m^p B_m^q - A_{m+1}^p B_{m+1}^q) \\
&= \sum_{m=0}^{\infty} C_m A_m^p (B_m^q - B_{m+1}^q) + \sum_{m=0}^{\infty} C_m (A_m^p - A_{m+1}^p) B_{m+1}^q \\
&\le \sum_{m=0}^{\infty} q C_m B_m^{q-1} A_m^p b_m + \sum_{m=0}^{\infty} C_m B_m^{q-1} (A_{m+1}^p - A_m^p) B_{m+1} \\
&\le \big(q \sup_m C_m B_m^{q-1}\big) \cdot \big(\sum_{m=0}^{\infty} A_m^p b_m + \sum_{m=0}^{\infty} (A_{m+1}^p - A_m^p) B_{m+1}\big) \\
&= \big(q K^{1/(p-1)}\big) \cdot \big(\sum_{m=0}^{\infty} A_m^p b_m + \sum_{m=1}^{\infty} (A_m^p - A_0^p) b_m\big) \\
&\le \big(2q K^{1/(p-1)}\big) \cdot \big(\sum_{m=0}^{\infty} A_m^p b_m\big).
\end{aligned}$$

Combining the above inequality with (1) yields

$$\sum_{m=0}^{\infty} A_m^p b_m \le p^p (2q)^{p-1} K \sum_{m=0}^{\infty} \frac{a_m^p}{c_m^{p-1}}.$$

Taking limits proves the lemma. $\square$

Theorem 3.5. *If Ω is a b-strip, $h_0 \in \Omega$ and Δ_0 is the disk $D(h_0, \rho_0)$ with $\rho_0 = \frac{1}{2}\delta_\Omega(h_0)$, then*

$$N_{p,\Omega}^p(\Delta_0) \lesssim V_{p,\Omega}(h_0) + b^p \frac{m_2(\Omega)}{m_2(\Delta_0)}.$$

Proof. Let Ω be a b-strip and W a Whitney decomposition of Ω into dyadic squares. For each square $Q \in W$ we define

$$R(Q) = \{w \in \Omega : \Re w = \Re z, \text{ some } z \in Q\}$$

so that $R(Q)$ is the portion of Ω in the vertical strip determined by Q. Since Ω is a b-strip it follows that so is each $R(Q)$.

Let γ be a centerline for Ω as defined prior to the proof of Theorem 2.7. Let $W(\gamma)$ denote the squares $Q \in W$ that contain a point of γ. Given two squares $Q_1, Q_2 \in W(\gamma)$, observe, from the properties of dyadic intervals, that if $R(Q_1) \cap R(Q_2)$ is nonempty then either

$$R(Q_1) \subset R(Q_2) \quad \text{or} \quad R(Q_2) \subset R(Q_1).$$

Thus, the subcollection $\{R(Q)\}$, where $Q \in W(\gamma)$ and $R(Q)$ is maximal with respect to set inclusion, gives a decomposition of Ω. Using the natural ordering obtained from their real parts we see that there is a doubly infinite sequence of squares $\{Q_j\}_{j=-\infty}^{\infty}$ in W satisfying:

(a) $Q_j \cap \gamma \neq \emptyset$ for all j,
(b) $\Omega = \bigcup R(Q_j)$,
(c) $R(Q_i) \cap R(Q_j) = \emptyset$ whenever $i \neq j$,
(d) if $i < j$, $w_i \in R(Q_i)$ and $w_j \in R(Q_j)$, then $\Re w_i < \Re w_j$, and
(e) $h_0 \in R(Q_0)$.

Now let $Q \in W$. If $Q \cap \gamma$ is nonempty, then our construction guarantees that $Q \subset R(Q_m)$ for some m and hence $d(Q) \leq d(Q_m)$. Now consider the case that $Q \cap \gamma = \emptyset$; we prove that $d(Q)$ can not be arbitrarily large compared to $d(Q_m)$:

(f) If $Q \in W$ intersects $R(Q_m)$, then $d(Q) \lesssim d(Q_m)$.

As in the proof of Theorem 2.7, let $f : \mathbb{D} \to \Omega$ be a conformal map which maps $(-1, 1)$ onto γ with $\Re f(t) \to \beta$ as $t \to 1$. Let $w_m \in \gamma \cap Q_m$ and put $\delta_m = \delta_\Omega(w_m)$. By Lemma 2.3 there is a hyperbolic geodesic crosscut τ_m which is almost orthogonal to γ at w_m and satisfies $\Lambda(\tau_m) \approx \delta_m$. Since τ_m is a crosscut, it has endpoints $\zeta_m^+, \zeta_m^- \in \partial\Omega$; where ζ_m^+ corresponds (under the map f) to a point of $\partial\mathbb{D}$ with positive imaginary part. As before, we know that

$$Z_m^+ = \{w : \Re w = \Re\zeta_m^+, \Im w \geq \Im\zeta_m^+\} \text{ and } Z_m^- = \{w : \Re w = \Re\zeta_m^-, \Im w \leq \Im\zeta_m^-\}$$

are contained in the complement of Ω.

Suppose that $d(Q)$ is very large compared to $d(Q_m)$. Since $Q \in W$ there are adjacent Whitney squares of comparable size on the left and right of Q which we denote by Q_- and Q_+. Together, these squares project onto an interval I on the real axis of size comparable to $d(Q)$, with the point w_m projecting to a point which is approximately in the middle of this interval. Now consider the curve $Z_m^+ \cup \tau_m \cup Z_m^-$. If $d(Q)$ is sufficiently large, then this curve must project onto a subinterval of I. Since $Z_m^+ \cup Z_m^-$ is contained in the complement of Ω, the only way this can happen is for ζ_m^+ to be above the set $Q_- \cup Q \cup Q_+$ and for ζ_m^- to be below. But then $\Lambda(\tau_m) \gtrsim d(Q)$ which contradicts our assumption. This proves property (f).

Fix $u \in C^1(\Omega)$ and assume that u vanishes on Δ_0. Let's denote $R(Q_j)$ by R_j. Then by applying Lemma 3.3 to R_j we have

$$\begin{aligned}\int\limits_{R_j} |u|^p &\lesssim \int\limits_{R_j} |u - u_{Q_j}|^p \, dm_2 + |u_{Q_j} - u_{Q_0}|^p m_2(R_j) + |u_{Q_0}|^p m_2(R_j) \\ &\lesssim b^p \int\limits_{R_j} |\nabla u|^p + |u_{Q_j} - u_{Q_0}|^p m_2(R_j) + |u_{Q_0}|^p m_2(R_j).\end{aligned}$$

Summing on j we see that

$$\int\limits_{\Omega} |u|^p \lesssim b^p \int\limits_{\Omega} |\nabla u|^p + \Sigma_2 + \Sigma_3$$

where $\Sigma_3 = |u_{Q_0}|^p m_2(\Omega)$. Thus, we need to estimate these last two summands in terms of the gradient integral.

There is a Whitney square Q which contains h_0. Since $h_0 \in R_0$ property (f) yields that $d(Q) \lesssim d(Q_0)$ and hence $\rho_0 \lesssim d(Q_0)$. Thus, there is a point $\widetilde{h}_0$ and a disk $\widetilde{\Delta}_0$ with the following properties: $\widetilde{\Delta}_0 \subset \Delta_0 \cap R_0$ and $m_2(\Delta_0) \approx m_2(\widetilde{\Delta}_0)$. By Lemma 3.1, Lemma 3.3 and the remark following it, we deduce that

$$\begin{aligned} |u_{Q_0}|^p m_2(R_0) &\lesssim \int\limits_{R_0} |u - u_{Q_0}|^p + \int\limits_{R_0} |u|^p \\ &\lesssim b^p \int\limits_{R_0} |\nabla u|^p + N^p_{p,\Delta_0}(R_0) \int\limits_{R_0} |\nabla u|^p \\ &\lesssim \Big(\frac{m_2(R_0)}{m_2(\Delta_0)}\Big) b^p \int\limits_{R_0} |\nabla u|^p. \end{aligned}$$

Thus,

$$\Sigma_3 \lesssim \Big(\frac{m_2(\Omega)}{m_2(\Delta_0)}\Big) b^p \int\limits_{R_0} |\nabla u|^p.$$

and we have now reduced the proof of the theorem to establishing the following inequality:

$$\Sigma_2 = \sum |u_{Q_j} - u_{Q_0}|^p m_2(R_j) \lesssim \big(V_{p,\Omega}(w_0) + b^p \frac{m_2(\Omega)}{m_2(\Delta_0)}\big) \int\limits_{\Omega} |\nabla u|^p. \tag{2}$$

Fix a positive integer m and consider the geometry of the squares Q_m and Q_{m+1}. Let Q'_{m+1} denote a square in W which is adjacent and to the left of Q_{m+1}. Since Q'_{m+1} has comparable size to Q_{m+1} and clearly intersects R_m by construction, we see that $d(Q_{m+1}) \lesssim d(Q_m)$ by property (f). We obtain a similar relationship for Q_m and hence $d(Q_{m+1}) \approx d(Q_m)$ for all m. Since $w_m \in Q_m$ we have $\delta_m \approx d(Q_m)$.

Now, Ω is a b-strip and hence there is a rectangle, contained in R_m, with dimensions comparable to $\delta_m \times \eta_m$ connecting Q_m to Q'_{m+1}; where $\eta_m = |w_{m+1} - w_m|$. Let T_m denote the union of this rectangle with $Q_m \cup Q'_{m+1}$. In order to apply Lemma 3.2, we may view T_m as a chain of comparably sized Whitney cubes connecting Q_m to Q'_{m+1}.

By Lemma 3.2 and Hölder's inequality we see that for $1 < p < \infty$

$$\begin{aligned} |u_{Q_m} - u_{Q_0}| &\le \sum_{j=1}^{m} |u_{Q_j} - u_{Q_{j-1}}| \lesssim \sum_{j=1}^{m} \frac{1}{d(Q_j)} \int\limits_{Q_j \cup T_{j-1}} |\nabla u| \\ &\lesssim \sum_{j=0}^{m} \frac{1}{d(Q_j)} \int\limits_{T_j} |\nabla u| \le \sum_{j=0}^{m} m_2(T_j)^{\frac{p-1}{p}} d(Q_j)^{-1} \Big(\int\limits_{T_j} |\nabla u|^p\Big)^{\frac{1}{p}}. \end{aligned}$$

Hence,

$$\sum_{m=0}^{\infty} |u_{Q_m} - u_{Q_0}|^p m_2(R_m) \lesssim \sum_{m=0}^{\infty} \Big(\sum_{j=0}^{m} a_j\Big)^p b_m \tag{3}$$

where for $m = 0, 1, \dots$ we have

$$a_m = m_2(T_j)^{(p-1)/p} d(Q_j)^{-1} \Big(\int_{T_m} |\nabla u|^p \Big)^{1/p} \quad \text{and} \quad b_m = m_2(R_m).$$

In order to apply Lemma 3.4 we define

$$c_m = m_2(T_m) d(Q_m)^{-p/(p-1)} \approx \frac{\eta_m}{\delta_m^{1/(p-1)}} \tag{4}$$

and then from (3) we have (using the notation of the lemma)

$$\sum_{m=0}^{\infty} |u_{Q_m} - u_{Q_0}|^p m_2(R_m) \lesssim \big(\sup_m C_m^{p-1} B_m\big) \sum_{m=0}^{\infty} \int_{T_m} |\nabla u|^p \, dm_2.$$

Hence (2) will follow from the above and the proof will be complete if we can obtain the appropriate bound for $\sup C_m^{p-1} B_m$.

To do this we must examine the geometry of the quantities C_m and B_m. Of course, the geometry of B_m is clear since

$$B_m = \sum_{j=m}^{\infty} m_2(R_j) = m_2(R_m \cup R_{m+1} \cup \cdots).$$

It will be convenient for us to replace B_m by B_{m+2}. We claim that the theorem will be proved once we establish that

$$C_m^{p-1} B_{m+2} \lesssim V_{p,\Omega}(h_0) + b^p \frac{m_2(\Omega)}{m_2(\Delta_0)} \tag{5}$$

holds for all $m = 0, 1, \dots$. This follows since $b_m \le b d(Q_m)$ and hence

$$\begin{aligned} C_m^{p-1} B_m &= (C_{m-1} + c_m)^{p-1} (b_m + B_{m+1}) \\ &\lesssim C_{m-1}^{p-1} B_m + C_m^{p-1} B_{m+1} + c_m^{p-1} b_m \\ &\lesssim \sup_n C_n^{p-1} B_{n+1} + b^p. \end{aligned}$$

Repeating this argument with $C_m^{p-1} B_{m+1}$ replacing the left hand side above gives the desired reduction.

Small values of m are easy to handle in (5). Recall that Q_m and Q_{m-1} have comparable size and hence there is a $\lambda > 1$ satisfying $d(Q_m) \ge d(Q_{m-1})/\lambda$ and $\delta_m \approx d(Q_m)$. Hence

$$\begin{aligned} C_m^{p-1} \cdot B_{m+2} &\lesssim \Big(\frac{b}{\delta_0^{1/(p-1)}} + \cdots + \frac{b}{\delta_m^{1/(p-1)}} \Big)^{p-1} m_2(\Omega) \\ &\lesssim \big(1 + \lambda^{1/(p-1)} + \cdots + \lambda^{m/(p-1)}\big)^{p-1} \frac{b^{p-1}}{\delta_0} m_2(\Omega) \\ &\lesssim (m+1)^{p-1} \lambda^m \cdot b^p \frac{m_2(\Omega)}{m_2(\Delta_0)}. \end{aligned}$$

Thus, it suffices to prove (5) for m sufficiently large.

Next, we claim that there is an m_0 depending only on the Whitney decomposition so that

$$C_{m-1} \lesssim \int\limits_{\gamma(w_0,w_m)} \frac{ds}{\delta_\Omega^{1/(p-1)}} \approx k_p(w_0,\tau_m) \tag{6}$$

whenever $m \geq m_0$.

To prove this recall that $w_m \in Q_m \cap \gamma$ and that we have constructed the hyperbolic geodesic τ_m and vertical rays Z_m^+ and Z_m^-. Let $\gamma_m = \gamma(w_{m-1}, w_m)$. Now suppose that $w \in \gamma_m$. Then w belongs to the region bounded by the two curves $Z_j^+ \cup \tau_j \cup Z_j^-$ $(j = m-1, m)$. The proof of property (f) makes it clear that $\delta_\Omega(w) \lesssim \delta_m$. Since $\eta_m = |w_{m+1} - w_m|$ it follows that $\eta_m \leq \Lambda(\gamma_{m+1})$ and hence,

$$C_{m-1} \approx \sum_{j=0}^{m-1} \frac{\eta_j}{\delta_j^{1/(p-1)}} \lesssim \sum_{j=1}^{m} \int_{\gamma_j} \frac{ds}{\delta_\Omega^{1/(p-1)}} \leq \int\limits_{\gamma(w_0,w_m)} \frac{ds}{\delta_\Omega^{1/(p-1)}}.$$

Since the hyperbolic diameter of any Whitney square is comparable to one, there is an $m_0 = m_0(W)$ so that $\rho_\Omega(w_0, w_m) \geq 1$ holds for all $m \geq m_0$. Now (6) follows from Lemma 2.6.

Let $L_m = \Omega_{x_m}$ where $x_m = \Re w_m$.

Claim. For $m \geq m_0$ we have

$$C_{m-1}^{p-1} B_{m+1} \lesssim k_p^{p-1}(w_0,\tau_m) m_2(\Omega(L_m)) \lesssim V_{p,\Omega}(h_0) + b^p \frac{m_2(\Omega)}{m_2(\Delta_0)}. \tag{7}$$

The first relation follows immediately from (6). Let us observe that there is a curve from h_0 to w_0 of length $\lesssim b$ along which $\delta_\Omega \gtrsim \rho_0$. Hence, by definition

$$k_p^{p-1}(w_0, h_0) \lesssim b^{p-1}/\rho_0. \tag{8}$$

Now let $\widetilde{w}_m$ be the point on $\gamma(w_0, w_m)$ at which

$$\int_{\gamma(w_0,\widetilde{w}_m)} \frac{ds}{\delta_\Omega^{1/(p-1)}} = \int_{\gamma(\widetilde{w}_m,w_m)} \frac{ds}{\delta_\Omega^{1/(p-1)}}$$

and let $\widetilde{\tau}_m$ be a hyperbolic geodesic crosscut through $\widetilde{w}_m$ which is nearly orthogonal to γ and has length comparable to $\delta_\Omega(\widetilde{w}_m)$. By modifying m_0, if necessary, we assume that $\rho_\Omega(w_0, \widetilde{w}_m) \geq 1$ so that

$$k_p(w_0, \widetilde{\tau}_m) \approx \int_{\gamma(w_0,\widetilde{w}_m)} \frac{ds}{\delta_\Omega^{1/(p-1)}}.$$

Case 1. $\widetilde{\tau}_m \cap L_m = \emptyset$:

We show that this implies that $\Omega(L_m) \subset \Omega(\widetilde{\tau}_m)$. Suppose to the contrary that $\widetilde{w}_m \in \Omega(L_m)$. Then there is a Jordan region contained in Ω and bounded by a subarc of γ containing $\widetilde{w}_m$ and a subarc of L_m. By the Jordan curve theorem it follows that $\widetilde{\tau}_m$, which clearly contains a point in the interior of this Jordan region by construction, must intersect the boundary of this region at a point other than $\widetilde{w}_m$. Since this point must be in L_m we have a contradiction to our hypothesis. This proves that $\Omega(L_m) \subset \Omega(\widetilde{\tau}_m)$ and hence that $k_p(w_0, L_m) \geq k_p(w_0, \widetilde{\tau}_m)$. It follows from this, the triangle inequality for k_p and (8) that

$$\begin{aligned} k_p^{p-1}(w_0, \tau_m) m_2(\Omega(L_m)) &\lesssim k_p^{p-1}(w_0, L_m) m_2(\Omega(L_m)) \\ &\lesssim \big(k_p(h_0, L_m) + \frac{b}{\rho_0^{1/(p-1)}}\big)^{p-1} m_2(\Omega(L_m)) \\ &\lesssim k_p^{p-1}(h_0, L_m) m_2(\Omega(L_m)) + b^p \frac{m_2(\Omega)}{m_2(\Delta_0)}; \end{aligned}$$

which implies (7).

Case 2. $\widetilde{\tau}_m \cap L_m$ is nonempty:

As argued earlier in the proof of Theorem 2.7, this means there is a point $w_m^\star \in \gamma(\widetilde{w}_m, w_m)$ with the property that $k_p(w_0, w_m^\star) \lesssim b\delta_\Omega(w_m^\star)^{-1/(p-1)}$. Let $\tau_m^\star$ be a hyperbolic geodesic which is nearly orthogonal to γ at $w_m^\star$ with length comparable to $\delta_\Omega(w_m^\star)$. If $\tau_m^\star \cap L_m = \emptyset$ then (7) follows as in case 1. Otherwise, there is a vertical crosscut $L_m^\star$ with the property that $\Omega(L_m^\star) \subset \Omega(\tau_m^\star)$ and that the set $\Omega(L_m) \setminus \Omega(L_m^\star)$ is contained in a vertical strip of width comparable to $\delta_\Omega(w_m^\star)$. Hence,

$$\begin{aligned} k_p^{p-1}(w_0, \tau_m) m_2(\Omega(L_m)) &\lesssim k_p^{p-1}(w_0, \tau_m^\star)\big(m_2(\Omega(L_m^\star)) + b\delta_\Omega(w_m^\star)\big) \\ &\lesssim k_p^{p-1}(w_0, L_m^\star) m_2(\Omega(L_m^\star)) + b^p \end{aligned}$$

and (7) follows as before.

The proof of the theorem is now complete. □

4. The necessity of the $K_{p,\Omega}$ Condition

In this section we prove Theorem B in the introduction. For a given hyperbolic geodesic we need only construct an appropriate function in the Sobolev space $W^{1,p}(\Omega)$ and apply the Poincaré inequality. We first give a simplified treatment of this construction.

Fix a point $w_0 \in \Omega$ and a hyperbolic geodesic τ which does not contain w_0. Put $\delta_0 = \delta_\Omega(w_0)$. Let γ be the hyperbolic geodesic in Ω which contains w_0 and is orthogonal to τ and let $w_\tau \in \tau$ be the point where these curves intersect. Our strategy is to decompose Ω in a manner similar to the decomposition in Lemma 2.6. We assume that n is a positive integer satisfying $2n+1 < \rho_\Omega(w_0, w_\tau) \leq 2n+3$. We denote by $w_1, \ldots, w_{2n}$ the points on $\gamma(w_0, w_\tau)$ determined by the relation $\rho_\Omega(w_0, w_m) = m$ and we put $\delta_m = \delta_\Omega(w_m)$. As before, we let τ_m be a hyperbolic geodesic which is nearly orthogonal to γ at w_m and has length comparable to δ_m. By Lemma 2.3, the τ_m's are disjoint and $\Omega(\tau) \subset \Omega(\tau_{2n})$.

Define a function u_0 on γ by setting $u_0(w) = 0$ if w precedes w_1, for $w \in \gamma(w_1, w_{2n})$ we take

$$u_0(w) = \int\limits_{\gamma(w_1,w)} \frac{ds}{\delta_\Omega^{1/(p-1)}}$$

and finally, $u_0(w) = u_0(w_{2n})$ for all $w \in \gamma \cap \Omega(\tau_{2n})$. It is elementary that u_0 is continuously defined on γ.

Let R_m denote the simply connected subset of Ω between the crosscuts τ_m and τ_{m-1}. As before let $\gamma_m = \gamma(w_{m-1}, w_m)$. Suppose that there exists a function u defined on Ω which satisfies the following properties

(a) $u \in W^{1,p}(\Omega) \cap C(\Omega)$,
(b) u is constant on $\Omega(\tau_{2n})$ and hence on $\Omega(\tau)$, with $u \approx k_p(w_0, \tau)$,
(c) u is zero on the component of $\Omega \setminus \tau_1$ containing w_0 and
(d) For $m = 2, \ldots, 2n$, we have

$$\int\limits_{R_m} |\nabla u|^p \lesssim \int\limits_{\gamma_m} \frac{ds}{\delta_\Omega^{1/(p-1)}}.$$

Remark. We note that the function u above is relevant to Maz'ja's capacitary theory of p-Poincaré domains. It would essentially determine the capacity of the condenser pair $(\Omega \setminus \Omega(\tau_1), \Omega(\tau_{2n}))$, which is an integral part of his characterization. See §4 in [Maz85].

Theorem 4.1. *Suppose that Ω is a p-Poincaré domain and $w_0 \in \Omega$. If for every hyperbolic geodesic τ with $\rho_\Omega(w_0, \tau) \geq 3$ there exists a function u defined on Ω satisfying the above properties, then*

$$K_{p,\Omega}(w_0) \lesssim \frac{m_2(\Omega)}{\delta_0^2} M_p^p(\Omega). \tag{1}$$

Proof. Fix a smooth function ϕ which is zero for $|z| \geq 1$, equals one for $|z| \leq 1/2$ and satisfies $0 \leq \phi \leq 1$. Let $\rho_0 = \delta_\Omega(w_0)/2$ and put $\Delta_0 = D(w_0, \rho_0) \subset \Omega$. Define v to be a *bump function* for Δ_0; i.e., let $v(w) = \phi((w - w_0)/\rho_0)$. Since $0 \leq v_\Omega \leq 1$, we see that $|v - v_\Omega| \geq 1/2$ on a set of area $\gtrsim \delta_0^2$. Hence,

$$m_2(\Delta_0) \lesssim \int\limits_\Omega |v - v_\Omega|^p \leq M_p^p(\Omega) \int\limits_\Omega |\nabla v|^p \lesssim M_p^p(\Omega) \delta_0^{2-p}$$

and it follows that

$$\delta_\Omega(w_0) \lesssim M_p(\Omega) \tag{2}$$

for all $w_0 \in \Omega$.

Suppose first that $\rho_\Omega(w_0, w_\tau) \leq 3$. By Lemma 2.2, we have

$$\Lambda(\gamma(w_0, w_\tau)) \lesssim \delta_0 \quad \text{and} \quad \delta_\Omega(w) \gtrsim \delta_0,$$

for all $w \in \gamma(w_0, w_\tau)$. Hence by Lemma 2.6 and (2) we have

$$\begin{aligned} k_p^{p-1}(w_0,\tau)\cdot m_2(\Omega(\tau)) &\lesssim \left(\int\limits_{\gamma(w_0,w_\tau)} \frac{ds}{\delta_\Omega^{1/(p-1)}} \right)^{p-1} m_2(\Omega) \\ &\lesssim \delta_0^{p-2} m_2(\Omega) \lesssim \frac{m_2(\Omega)}{\delta_0^2} M_p^p(\Omega). \end{aligned}$$

which proves (1) in this case. Hence, it suffices to assume that the hyperbolic distance between w_0 and τ is at least 3.

Now assume that τ is a hyperbolic geodesic with $2n+3 \geq \rho_\Omega(w_0,\tau) > 2n+1 \geq 3$ and that u satisfies the properties described above. We may need to modify the definition of ρ_0 slightly so that $\Delta_0 \cap \tau_1 = \emptyset$. By Lemma 2.2 this can be done with $\rho_0 \approx \delta_0$. Since $\rho_\Omega(w_0, w_\tau) \leq 2n+3$ it follows from Lemma 2.2 and Lemma 2.6 that

$$k_p(w_0,\tau) \approx k_p(w_0,w_{2n}) \approx k_p(w_1,w_{2n}) \approx \int\limits_{\gamma(w_1,w_{2n})} \frac{ds}{\delta_\Omega^{1/(p-1)}}. \tag{3}$$

By property (a) the Poincaré inequality holds for the function u. Since $u = 0$ on Δ_0 we use the corresponding inequality with $N_{p,\Omega}(\Delta_0)$ in place of $M_p(\Omega)$ and apply Lemma 3.1. Combining this with property (b) and (3) we get

$$\begin{aligned} k_p^p(w_0,\tau)\cdot m_2(\Omega(\tau)) &\approx \int\limits_{\Omega(\tau)} |u|^p \\ &\leq \int\limits_{\Omega} |u|^p \leq N_{p,\Omega}^p(\Delta_0) \int\limits_{\Omega} |\nabla u|^p \\ &\lesssim \frac{m_2(\Omega)}{m_2(\Delta_0)} M_p^p(\Omega) \int\limits_{\Omega} |\nabla u|^p \end{aligned}$$

and on the other hand properties (b), (c) and (d) yield that

$$\begin{aligned} \int\limits_{\Omega} |\nabla u|^p &= \sum_{m=2}^{2n} \int\limits_{R_m} |\nabla u|^p \\ &\lesssim \sum_{m=2}^{2n} \int\limits_{\gamma_m} \frac{ds}{\delta_\Omega^{1/(p-1)}} = \int\limits_{\gamma(w_1,w_{2n})} \frac{ds}{\delta_\Omega^{1/(p-1)}} \approx k_p(w_0,\tau). \end{aligned}$$

Combining these inequalities gives

$$k_p^{p-1}(w_0,\tau)\cdot m_2(\Omega(\tau)) \lesssim \frac{m_2(\Omega)}{\delta_0^2} M_p^p(\Omega)$$

which proves the theorem. $\square$

We have now reduced the proof of Theorem B to the construction of a certain continuous function on R_m with an estimate on the integral of its gradient. Of course, we must also show that these functions combine to form a Sobolev function on Ω but this will involve a standard result in the theory of these functions. The curve γ divides R_m into two pieces which we denote by R_m^+ for the top half; which is the image of points with positive imaginary parts under a conformal mapping of $\mathbb{D}$, and R_m^- for the bottom half. We only describe the construction of u on R_m^+ since the other construction is similar.

Observe that $\partial R_m^+ \cap \Omega$ consists of three curves; namely, τ_{m-1}^+, γ_m and τ_m^+ and also observe that at the two "vertices" w_{m-1}, w_m these curves meet at nearly a right angle. The basic idea of the construction is to prove that there is a bilipschitzian change of variables on $\mathbb{C}$ (with bounds on the distortion which are independent of m) which maps these three curves onto three sides of a square of side length δ_m. Once this is done, a function will be defined on this square and transferred back to R_m^+ by means of this change of variables.

We already know that the lengths of these three curves are all comparable to δ_m by construction and that consecutive curves are nearly orthogonal as observed above. We have two problems to deal with: one is to control the smoothness of the curves (especially near $\partial\Omega$) and the other is that the endpoints ζ_{m-1}^+, ζ_m^+ of τ_{m-1}^+, τ_m^+ may be very close together in the Euclidean metric.

We need to modify the curves τ_{m-1}^+, τ_m^+ near $\partial\Omega$. Fix a small positive number $\eta < 1/2$. We will reduce the value of η at various points in the proof. There is a first point a_m^+ on the curve τ_m^+, starting from w_m, at which $\delta_\Omega(a_m^+) = \eta\delta_m$. Since $|\zeta_m^+ - w_m| \geq \delta_m$, $a_m^+ \neq w_m$. Let $b_m^+ \in \partial\Omega$ be any boundary point which satisfies $|a_m^+ - b_m^+| = \eta\delta_m$ and denote by $[a_m^+, b_m^+]$ the line segment with these endpoints. Now denote by σ_m^+ the curve $\tau_m^+(w_m, a_m^+)$ followed by $[a_m^+, b_m^+]$. Since $b_m^+ \in \partial\Omega$ we obtain a new crosscut σ_m through w_m by combining σ_m^+ and σ_m^-.

Lemma 4.2. *Let σ be a hyperbolic geodesic arc of Ω with endpoints w, $w' \in \Omega$. If $\rho_\Omega(w, w') \leq \beta$, then*
$$\Lambda(\sigma) \leq 4e^{3\beta}|w - w'|.$$

Proof. There is a conformal mapping f of $\mathbb{D}$ onto Ω which maps the origin to w and a point r with $0 < r < 1$ onto w'. Since σ is a hyperbolic geodesic, the line segment $[0, r]$ is mapped onto the arc σ. Since
$$\log \frac{1+r}{1-r} = \rho_{\mathbb{D}}(0, r) = \rho_\Omega(w, w') \leq \beta$$
it follows from the distortion theorem (see Theorem 1.3, Corollary 1.4 in [Pom92]) that
$$|w - w'| \geq \frac{r}{4}|f'(0)| \quad \text{and} \quad |f'(t)| \leq |f'(0)|e^{3\beta}$$
for all t satisfying $0 \leq t \leq r$. Hence
$$\Lambda(\sigma) = \int_0^r |f'(t)|\, dt \leq re^{3\beta}|f'(0)| \leq 4e^{3\beta}|w - w'|$$
which proves the lemma. $\square$

Lemma 4.3. *There is a constant c_η depending only on η such that for all $m = 1, \dots, 2n$ we have*

$$\Lambda(\sigma_m^+(w, w')) \leq c_\eta |w - w'| \tag{4}$$

for all $w, w' \in \sigma_m^+$.

Proof. Let us estimate $\rho_\Omega(w_m, a_m^+)$ using the quasi-hyperbolic metric k_2:

$$k_2(w_m, a_m^+) \leq \int\limits_{\tau_m^+(w_m, a_m^+)} \frac{ds}{\delta_\Omega} \leq \frac{\Lambda(\tau_m)}{\eta\delta_m} \lesssim \frac{1}{\eta}$$

and thus $\rho_\Omega(w_m, a_m^+) \lesssim \eta^{-1}$. Applying Lemma 4.2 we see that (4) holds provided $w, w' \in \tau_m^+$.

Of course, if $w, w' \in [a_m^+, b_m^+]$ then (4) is trivially true with constant 1. Finally, suppose that $w \in \tau_m^+(w_m, a_m^+)$ and that $w' \in [a_m^+, b_m^+]$. Observe that the definitions of a_m^+ and b_m^+ imply that $\tau_m^+(w_m, a_m^+)$ is disjoint from the disk $D(b_m^+, \eta\delta_m)$. It follows from this that $|w - w'| \geq |a_m^+ - w'|$ and the rest of the argument is straightforward. □

Now consider the disk $D_m^+ = D(a_m^+, \eta\delta_m)$. Our construction satisfies the hypothesis of Lemma 2.5 and hence: if σ is any curve in Ω with endpoints a_{m-1}^+ and a_m^+ then $\Lambda(\sigma) \gtrsim \delta_{m-1}$. Suppose $D_{m-1}^+ \cap D_m^+$ is nonempty. Then the line segment $[a_{m-1}^+, a_m^+] \subset \Omega$ and the above implies that its length is $\gtrsim \delta_{m-1}$. On the other hand, its length is less than $\eta(\delta_{m-1} + \delta_m)$. Since $\delta_{m-1} \approx \delta_m$ it follows that η can be chosen so that

$$D_i^+ \cap D_j^+ = \emptyset \quad \text{whenever} \quad i \neq j. \tag{5}$$

Lemma 4.4. *There exists $\epsilon > 0$ with the following property: If $\{D_i\}$ is a disjoint collection of three open disks of radius r and if $\{z_i\}$ are points in ∂D_i, where $1 \leq i \leq 3$, then there is a point z_i satisfying*

$$|z_1 - z_i| > \epsilon r\,.$$

Proof. An elementary geometric argument yields this result.

Corollary 4.5. *There is an $\epsilon > 0$ with the following property: Given i with $2(k-1) < i \leq 2k$, there is a j with $2k < j \leq 2(k+1)$ satisfying*

$$D(b_i^+, \epsilon\eta\delta_i) \cap D(b_j^+, \epsilon\eta\delta_j) = \emptyset.$$

Proof. Since the number $\delta_{2(k-1)}, \dots, \delta_{2(k+1)}$ are comparable the proof is immediate from Lemma 4.4. □

We are now in a position to modify our decomposition of Ω. By Corollary 4.5, we can fix ϵ and chose exactly n of the original crosscuts σ_m, which we shall relabel as $\sigma_1, \dots, \sigma_n$, along with the corresponding w_m's, a_m^+'s, etc., so that the following three properties are fulfilled:

(a) $1 \leq \rho_\Omega(w_{m-1}, w_m) < 4$ for all $m = 1, \dots, n$,
(b) $D(b_{m-1}^+, \epsilon\eta\delta_{m-1}) \cap D(b_m^+, \epsilon\eta\delta_m) = \emptyset$ for $m = 2, \dots, n$ and
(c) $D_{m-1}^+ \cap D_m^+ = \emptyset$.

Continuing, we denote the region in Ω between σ_{m-1} and σ_m as R_m and suppose that there exists a function u defined on Ω which satisfies the following properties

(d) $u \in W^{1,p}(\Omega) \cap C(\Omega)$,
(e) u is constant on $\Omega(\tau)$ with $u \approx k_p(w_0, \tau)$,
(f) u is zero on the component of $\Omega \setminus \sigma_1$ containing w_0 and
(g) for $m = 2, \ldots, n$, we have

$$\int_{R_m} |\nabla u|^p \lesssim \int_{\gamma_m} \frac{ds}{\delta_\Omega^{1/(p-1)}}.$$

It follows that Theorem 4.1 is valid with this decomposition, and hence we have reduced the proof of Theorem B to proving that this modified function u exists.

Fix m with $2 \le m \le n$. We now construct the bilipschitzian square needed for our definition of u. Recall from the proof of Lemma 4.3 that the region $D(a_m^+, \eta\delta_m) \cap D(b_m^+, \eta\delta_m)$ is contained in Ω and is disjoint from $\tau_m^+(w_m, a_m^+)$. Since the line segment $[a_m^+, b_m^+]$ divides this region into two parts and since σ_m is a crosscut it follows that there is a point r_m^+ satisfying

(h) $r_m^+ \in R_{m+1} \cap D(a_m^+, \eta\delta_m) \cap D(b_m^+, \eta\delta_m)$,
(i) $|r_m^+ - b_m^+| = \frac{1}{2}\eta\delta_m$ and
(j) the angle between $[b_m^+, a_m^+]$ and $[b_m^+, r_m^+]$ is $\pi/4$.

Analogously there is a point l_m^+ satisfying

(h') $l_m^+ \in R_{m-1} \cap D(a_{m-1}^+, \eta\delta_{m-1}) \cap D(b_{m-1}^+, \eta\delta_{m-1})$,
(i') $|r_{m-1}^+ - b_{m-1}^+| = \frac{1}{2}\eta\delta_{m-1}$ and
(j') the angle between $[b_{m-1}^+, a_{m-1}^+]$ and $[b_{m-1}^+, r_{m-1}^+]$ is $\pi/4$.

Notice that the index refers to the region R_m and not to the crosscuts and in fact the notation is a reminder that for a fixed m, l_m^+ is to the left of R_m and r_m^+ is to the right.

Let us now define the curve μ_m^+ to be the curve that starts at l_m^+, goes straight to b_{m-1}^+, follows σ_{m-1}^+ to w_{m-1}, follows γ_m to w_m, follows σ_m^+ to b_m^+ and ends by going straight to r_m^+.

Lemma 4.6. *There is an $\eta > 0$ and a constant c_η depending only on η such that for all $m = 2, \ldots, n$ we have*

$$\Lambda(\mu_m^+(w, w')) \le c_\eta |w - w'| \tag{6}$$

for all $w, w' \in \mu_m^+$.

Proof. The proof is similar to the proof of Lemma 4.3. The curve μ_m^+ consists of five simpler curves for which (6) is either trivially valid or follows from Lemma 4.3. Thus, we must consider several cases where w is in one of these curves and w' is in another.

To simplify matters we first prove that unless the two curve segments are adjacent to each other (as components of μ_m) then the Euclidean distance between them is $\gtrsim \epsilon\eta\delta_m$. This will reduce the proof to examining adjacent curves since $\Lambda(\mu_m^+) \approx \delta_m$ and ϵ is fixed. By properties (b) and (c) we see that the Euclidean distance between any two nonadjacent linear segments is $\gtrsim \epsilon\eta\delta_m$, since $\delta_{m-1} \approx \delta_m$.

By Lemma 2.5 we may assume, after possibly reducing the value of η, that the Euclidean distance between σ_{m-1} and σ_m is least $\eta\delta_m$. Hence we are left with proving that γ_m is not too close to the straight-line segments. It will suffice to show that γ_m is disjoint from $D^+_{m-1} \cup D^+_m$. But we can obviously adjust the value of η so that this is the case since $\delta_\Omega(w) \approx \delta_m$ for all $w \in \gamma_m$ and each disk D_m has a point on its boundary which is in the complement of Ω. Thus, the Euclidean distance between nonadjacent curves is not too small, relative to δ_m.

Finally, we examine the interaction between adjacent curve segments. The adjacent linear segments are not a problem since the angle between them is $\pi/4$. Suppose now that $w \in \tau^+_{m-1}(w_{m-1}, a^+_{m-1})$ and $w' \in \gamma_m$. Let f be the conformal map of $\mathbb{D}$ onto Ω which maps $(-1,1)$ onto γ which maps the origin to w_{m-1} and satisfies $0 < f^{-1}(w_m) < 1$. By our construction, there is a point $z' > 0$ for which $[0, z']$ maps onto $\gamma(w_{m-1}, w')$. Let $z \in \mathbb{D}$ map onto w. By our construction, the diameter L of $\mathbb{D}$ through z maps onto τ_{m-1} and hence is nearly orthogonal to the real axis.

If $|w - w'| \geq \eta\delta_{m-1}$, then it is immediate that (6) holds because the curve has length comparable to δ_{m-1}. On the other hand, if $|w - w'| < \eta\delta_{m-1}$, then clearly for sufficiently small η we have

$$\rho_\Omega(w, w') \approx k_2(w, w') \approx \frac{|w - w'|}{\delta_\Omega(w')} \leq \frac{\eta\delta_{m-1}}{\delta_\Omega(w')}.$$

Hence we may assume that η has been chosen so that $\rho_\Omega(w, w') < 1$. But now $|w - w'| \approx |z - z'|\delta_{m-1}$ and the proof of (6) follows from the distortion theorem and the fact that L is nearly orthogonal to the real axis. $\square$

We now fix the value of η so that (6) holds. Our decomposition is complete and we next prove that our curves μ_m are bilipschitzianly equivalent to a segment of length δ_m, which is in turn equivalent to three sides of a square of side δ_m.

Lemma 4.7. *There is an absolute constant $\alpha > 0$ so that for each m, $2 \leq m \leq n$, there is a bilipschitzian homeomorphism T_m, of the entire plane $\mathbb{C}$, with the following properties*

(i) $\alpha^{-1}|z_1 - z_2| \leq |T_m(z_1) - T_m(z_2)| \leq \alpha|z_1 - z_2|$ *for all z_1, z_2 in $\mathbb{C}$,*
(ii) $T_m(\mu^+_m)$ *is contained in the boundary of the square*

$$S_m = \{(s,t) : 0 \leq s \leq \delta_m, \quad 0 \leq t \leq \delta_m\}$$

and lastly,
(iii) $T_m(l^+_m) = (0, \delta_m)$, $T_m(b^+_m) = (0, \delta_m/2)$, $T_m(w_{m-1}) = (0,0)$, $T_m(w_m) = (\delta_m, 0)$, $T_m(b^+_m) = (\delta_m, \delta_m/2)$ *and* $T_m(r^+_m) = (\delta_m, \delta_m)$.

Proof. Let $\psi_m : [0, \Lambda(\mu_m)] \to \mu_m$ be the arc length parametrization of μ_m. By (6) there is an absolute constant $\alpha_0 > 0$ such that $|s_2 - s_1| \leq \alpha_0|\psi_m(s_2) - \psi_m(s_1)|$ and trivially $|\psi_m(s_2) - \psi_m(s_1)| \leq |s_2 - s_1|$. Hence ψ_m is a Lipschitz embedding of the interval $[0, \Lambda(\mu_m)]$ onto μ_m. Since $\Lambda(\mu_m) \approx \delta_m$ we might as well assume that $\psi_m : [0, \delta_m] \to \mu_m$ is a Lipschitz embedding satisfying

$$\alpha_1^{-1}|s_2 - s_1| \leq |\psi_m(s_2) - \psi_m(s_1)| \leq \alpha_1|s_2 - s_1|$$

for some absolute constant α_1.

By Theorem B in [Tuk80] (see also [Tuk81], [JeKe] and Chapter 7.4 in [Pom92]), there is a homeomorphism Ψ_m of $\mathbb{C}$ whose restriction to the interval $[0, \delta_m]$ is ψ_m and which satisfies

$$\alpha_2^{-1}|z_2 - z_1| \leq |\Psi_m(z_2) - \Psi_m(z_1)| \leq \alpha_2|z_2 - z_1|$$

for all $z_1, z_2 \in \mathbb{C}$.

On the other hand, there is obviously a bilipschitz embedding ϕ_m of the interval $[0, \delta_m]$ onto the three sides of the square S_m with constant α_3. Using the above extension theorem again we get a bilipschitz homeomorphism Φ_m of $\mathbb{C}$ extending ϕ_m.

Put $T_m = \Phi_m \circ \Psi_m^{-1}$. Clearly, T_m satisfies properties (i) and (ii). In order to show that property (iii) can also be satisfied, we need only construct a bilipschitz embedding of the interval $[0, \delta_m]$ onto itself which maps $\Psi_m^{-1}(b_{m-1}^+)$ onto $\Phi_m^{-1}(0, \delta_m/2)$, $\Psi_m^{-1}(w_{m-1})$ onto $\Phi_m^{-1}(0, 0), \ldots.$ But this can clearly be done with a piecewise linear function with an absolute bound on the Lipschitz constant, since all the component curves of μ_m^+ have length comparable to δ_m. $\square$

We now define our function u which is to satisfy properties (d)-(g). First define $v_m(s,t)$ on S_m by

$$v_m(s,t) = (1 - \frac{s}{\delta_m})u_0(w_{m-1}) + \frac{s}{\delta_m}u_0(w_m)$$

whenever, $0 \leq s \leq \delta_m$ and $0 \leq t \leq \frac{\delta_m}{2}$, and

$$v_m(s,t) = (2 - \frac{2t}{\delta_m})v_m(s, \frac{\delta_m}{2}) + (\frac{2t}{\delta_m} - 1)\frac{u_0(w_{m-1}) + u_0(w_m)}{2}$$

when $0 \leq s \leq \delta_m$ and $\delta_m/2 \leq t \leq \delta_m$. Observe that $v_m(s, \delta_m)$ is independent of s and equals the average value $(u_0(w_{m-1}) + u_0(m))/2$. By the definition of u_0 and our construction we see that

$$u_0(w_m) - u_0(w_{m-1}) = \int\limits_{\gamma_m} \frac{ds}{\delta_\Omega^{1/(p-1)}} \approx \frac{\delta_m}{\delta_m^{1/(p-1)}}$$

and an elementary computation then shows that

$$\int\limits_{S_m} |\nabla v_m|^p \approx (\frac{u_0(w_m) - u_0(w_{m-1})}{\delta_m})^p \delta_m^2 \approx \frac{\delta_m}{\delta_m^{1/(p-1)}} \approx \int\limits_{\gamma_m} \frac{ds}{\delta_\Omega^{1/(p-1)}}. \tag{7}$$

Now we define $u(w) = v_m(T_m(w))$ provided $w \in T_m^{-1}(S_m) \cap R_m^+$ and $u(w) = (u_0(w_{m-1}) + u_0(m))/2$ whenever $w \in R_m^+ \setminus T_m^{-1}(S_m)$. We make the analogous definitions for points in R_m^- and of course we set $u = u_0(w_n)$ on $\Omega(\sigma_n)$ and $u = 0$ on the component of $\Omega \setminus \sigma_1$ containing w_0. It is easy to see that u is well defined and continuous on Ω and satisfies properties (e) and (f).

To simplify the discussion of properties (d) and (g) let us assume that the bilipschitzian homeomorphisms T_m are continuously differentiable. Then, property (i) of Lemma 4.7 would imply that the transformation T_m is approximately an isometry. Thus, the Jacobian $J_{T_m} \approx 1$ and by the chain rule $|\nabla u(w)| \approx |\nabla v_m(T_m(w))|$ whenever $w \in R_m \cap T_m^{-1}(S_m)$ and $\Im T_m(w) \neq \delta_m/2$. Hence the change of variables formula and (7) imply that

$$\begin{aligned}\int\limits_{R_m} |\nabla u|^p &= \int\limits_{R_m \cap T_m^{-1}(S_m)} |\nabla u|^p \approx \int\limits_{R_m \cap T_m^{-1}(S_m)} |(\nabla v_m) \circ T_m|^p J_{T_m} \\ &= \int\limits_{T_m(R_m) \cap S_m} |\nabla v_m|^p \lesssim \int\limits_{\gamma_m} \frac{ds}{\delta_\Omega^{1/(p-1)}}\end{aligned}$$

which yields property (g). To prove that u is a Sobolev function we first observe that it is piecewise differentiable. Secondly, u is a local Lipschitz function on Ω and hence it is absolutely continuous on any line segment contained in Ω. It follows from a well-known theorem in Sobolev space theory, see Theorem 2.1.4 in [Zie], that u has weak derivatives on Ω which agree with its classical derivative at almost all points of Ω. Thus, property (d) is fulfilled and we are done in this case.

To complete the proof in general we need to show that bilipschitzian transformations have classical derivatives almost everywhere and that the chain rule and change of variable formula are all valid. But these facts are well-known; see for example Chapter 2.2 [Zie]. This proves Theorem B in the following form:

Theorem 4.8. *If Ω is a simply connected domain with finite area, then*

$$K_{p,\Omega}(w_0) \lesssim \frac{m_2(\Omega)}{\delta_\Omega(w_0)^2} M_p^p(\Omega).$$

Theorem 4.8 can be used to produce additional useful lower bounds for $M_p(\Omega)$. We illustrate this with the following corollary and example. Given disjoint hyperbolic geodesics τ_1 and τ_2 in Ω, for $j = 1, 2$ we denote by $\Omega_j(\tau_1, \tau_2)$ the component of $\Omega \setminus \tau_j$ that is disjoint from $\tau_1 \cup \tau_2$ and denote by A_j its area.

Corollary 4.9. *Suppose τ_1 and τ_2 are disjoint hyperbolic geodesics in a simply connected domain Ω of finite area. Then, for $1 < p < \infty$,*

$$\frac{\max \delta_\Omega^2}{m_2(\Omega)} k_p^{p-1}(\tau_1, \tau_2) \min\{A_1, A_2\} \lesssim M_p^p(\Omega)\,.$$

Proof. Let $w_0 \in \Omega$ be such that $\delta_\Omega(w_0) \geq \max \delta_\Omega/2$ and suppose without loss of generality that $k_p(w_0, \tau_1) \geq k_p(w_0, \tau_2)$. Then $k_p(w_0, \tau_1) \geq k_p(\tau_1, \tau_2)/2$, by the triangle inequality. If $w_0 \in \Omega_1(\tau_1, \tau_2)$, then τ_1 would separate w_0 from τ_2, and so it would follow that $k_p(w_0, \tau_1) < k_p(w_0, \tau_2)$ in contradiction to our assumption. Thus

$w_0 \notin \Omega_1(\tau_1, \tau_2)$, and so $\Omega(\tau_1) = \Omega_1(\tau_1, \tau_2)$. Hence, by Theorem 4.8 and the above,

$$\begin{aligned} M_p^p(\Omega) &\gtrsim \frac{\delta_\Omega(w_0)^2}{m_2(\Omega)} K_{p,\Omega}(w_0) \\ &\geq \frac{\max \delta_\Omega^2}{4m_2(\Omega)} k_p^{p-1}(w_0, \tau_1) m_2(\Omega(\tau_1)) \\ &\geq \frac{\max \delta_\Omega^2}{4m_2(\Omega)} \frac{k_p^{p-1}(\tau_1, \tau_2)}{2^{p-1}} m_2(\Omega_1(\tau_1, \tau_2)) \,, \end{aligned}$$

which completes the proof.

Example 4.10. Let a and b be positive numbers with b/a large, and suppose Ω is a simply connected p-Poincaré domain containing the rectangle $R = (0, b) \times (0, a)$, and such that $[0, b] \times \{0, a\} \subset \partial\Omega$. Then, for $1 < p < \infty$,

$$b^p \lesssim \frac{m_2(\Omega)}{\max \delta_\Omega^2} M_p^p(\Omega) \tag{8}$$

and hence Ω can not contain arbitrarily long "rectangular passages".

To see this, let L_1 and L_2 be the vertical crosscuts of R corresponding to $x = b/3$ and $x = 2b/3$. These determine hyperbolic geodesics τ_1 and τ_2 using the prime ends of the L_j's. By the Gehring-Hayman theorem the lengths of these geodesics are comparable to a. If b/a is sufficiently large, then the Euclidean distance from τ_1 to τ_2 is comparable to b. Thus $k_p^{p-1}(\tau_1, \tau_2) \gtrsim b^{p-1}/a$. Also $m_2(\Omega_j(\tau_1, \tau_2)) \gtrsim m_2(R) = ab$, $j = 1, 2$ and so the desired inequality is an immediate consequence of Corollary 4.9.

5. Steiner Symmetrization

Pólya proved in 1948 [Pól] that the smallest positive eigenvalue for Laplace's operator, with Dirichlet boundary conditions, on a fixed domain Ω, will never decrease under Steiner symmetrization. This section is motivated by the corresponding question for Neumann boundary conditions. Indeed, $M_2^2(\Omega)$ equals the reciprocal of the smallest positive eigenvalue for the Neumann problem on Ω. See [DeLi], [Maz68], §4.10 in [Maz85] and §4 of [Sta] for more on this connection. Hence, Pólya's result suggests that $M_p(\Omega^\star) \leq M_p(\Omega)$. We shall see in Example 5.2 that this is false. There exists, for each $p > 1$, a p-Poincaré domain Ω for which $M_p(\Omega^\star) = \infty$. Thus, there exist domains for which the Neumann problem has a smallest positive eigenvalue whereas the corresponding problem on the Steiner symmetrized domain has arbitrarily small positive eigenvalues. Theorem 5.1 shows, however, that for b-strips Steiner symmetrization has a controlled effect on M_p. For more information on symmetrizations and eigenvalue problems in the theory of partial differential equations we cite [Bae], [Kaw], and the classical reference [PóSz] which contain many more references.

In this section we prove Theorem C in the following form.

Theorem 5.1. *Suppose that Ω is a b-strip with $M_p(\Omega) < \infty$ and let $w_0 \in \Omega$. Then*

$$M_p^p(\Omega^\star) \lesssim \frac{m_2(\Omega)}{\delta_\Omega(w_0)^2} \left(M_p^p(\Omega) + b^p \right).$$

Proof. Fix $w_0 \in \Omega$. For $w \in \Omega$, let $\pi_1(w)$ be the projection of w onto the real axis. Now it is geometrically obvious that

$$\delta_\Omega(w) \le \delta_{\Omega^\star}(\pi_1(w)) \quad (w \in \Omega) \tag{1}$$

because a disk centered on the real axis is a Steiner symmetric domain. Let γ be an arc in Ω from w_0 to w_1. Denote by x_0, x_1 their real parts and let $[x_0, x_1]$ be the line segment in $\Omega^\star$ with these endpoints. Assume for convenience that $x_0 < x_1$. By (1) we clearly have

$$\int_{x_0}^{x_1} \frac{dx}{\delta_{\Omega^\star}(x)^{1/(p-1)}} \le \int\limits_\gamma \frac{|dw|}{\delta_{\Omega^\star}(\pi_1(w))^{1/(p-1)}} \le \int\limits_\gamma \frac{ds}{\delta_\Omega^{1/(p-1)}}$$

and hence $k_{p,\Omega^\star}(x_0, x_1) \le k_{p,\Omega}(w_0, w_1)$. It follows that

$$V_{p,\Omega^\star}(x_0) \le V_{p,\Omega}(w_0) \tag{2}$$

because if L_x is the vertical crosscut of Ω corresponding to the real number $x \ne x_0$, then $m_2(\Omega(L_x)) = m_2(\Omega^\star(L_x^\star))$.

Let $\Delta_0^\star$ be the disk in $\Omega^\star$ which is centered at x_0 and has radius $\delta_{\Omega^\star}(x_0)/2$. Since $\Omega^\star$ is also a b-strip we can apply Theorem 3.5 along with Lemma 3.1 and (2) to deduce that

$$\begin{aligned} M_p^p(\Omega^\star) \lesssim N_{p,\Omega^\star}^p(\Delta_0^\star) &\lesssim V_{p,\Omega^\star}(x_0) + b^p \frac{m_2(\Omega^\star)}{\delta_{\Omega^\star}(x_0)^2} \\ &\lesssim V_{p,\Omega}(w_0) + b^p \frac{m_2(\Omega)}{\delta_\Omega(w_0)^2}. \end{aligned} \tag{3}$$

On the other hand, Theorem 2.7 and Theorem 4.8 yield

$$V_{p,\Omega}(w_0) \lesssim K_{p,\Omega}(w_0) + b^p \lesssim \frac{m_2(\Omega)}{\delta_\Omega(w_0)^2} M_p^p(\Omega) + b^p.$$

Finally, (3) and the above combine to prove the theorem. □

Example 5.2. Given $p > 1$, we construct a bounded simply connected p-Poincaré domain Ω whose Steiner symmetrization $\Omega^\star$ is not a p-Poincaré domain.

Case 1. $1 < p < 2$:

The domain Ω will be obtained by stacking together infinitely many rescaled versions of the domains $\Omega(n, a)$, with $n = 1, 2, \ldots$ and $0 < a < 1/n$, which is illustrated (along with its Steiner symmetrization) in Figure 1.

Each $\Omega(n, a)$ consists of a left square room and n right narrow rectangular rooms each joined to the left room by a passage of height a, except for the lowest right room which opens fully to the left room and has height a. The domain Ω is obtained as follows: Start off with a domain $\Omega(n_1, a_1)$. Next, adjacent to the lowest right room of $\Omega(n_1, a_1)$, we adjoin a rescaling of $\Omega(n_2, a_2)$ by removing the left boundary of this rescaled domain along with the right boundary of the lower right room of

$\Omega(n_1, a_1)$. The rescaled version of $\Omega(n_2, a_2)$ will have dimensions $1 \times a_1$. In the same fashion, we now adjoin to the lowest right room a rescaling of $\Omega(n_3, a_3)$ with dimensions $1/2 \times a_1 a_2$. Continuing this process indefinitely will produce a bounded domain Ω which depends on the sequences $\{a_i\}$ and $\{n_i\}$. We now proceed to show that these sequences can be chosen in such a way that $M_p(\Omega) < \infty$ while $M_p(\Omega^\star) = \infty$.

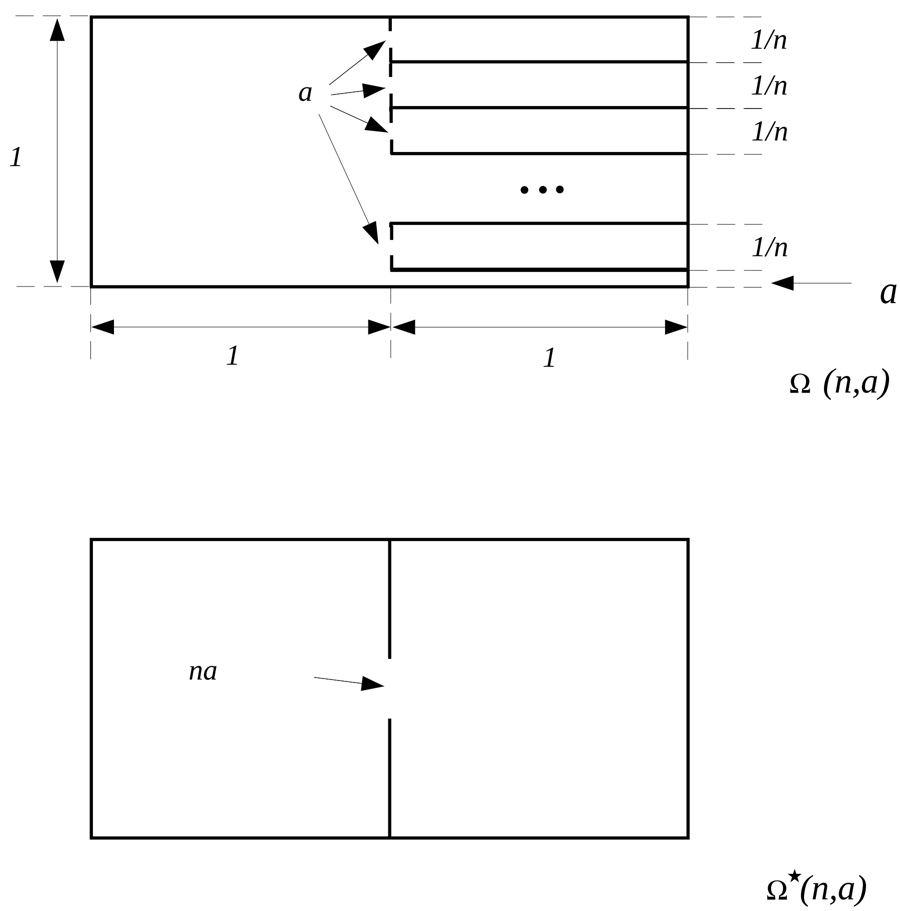

FIGURE 1. The domain $\Omega(n, a)$ and its Steiner symmetrization.

We first deal with Ω. We decompose Ω into its intersections with the horizontal strips determined by the partition of the vertical axis by the ordinates of the upper and lower boundaries of all the right rooms obtained from the rescaled $\Omega(n_i, a_i)$ which comprise Ω. We write

$$\Omega = \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{n_i - 1} \Omega_{i,j},$$

where for a given index i, the domains $\Omega_{i,j}$ $(1 \le j < n_i)$ correspond to the upper n_{i-1} right rooms of the rescaled $\Omega(n_i, a_i)$ part of Ω, taken in any order.

Let $\Delta_{i,j}$ denote the disk in $\Omega_{i,j}$ with diameter equaling the height of $\Omega_{i,j}$ and centered at the point $h_{i,j}$ with abscissa $x = 1/2$ (we introduce the usual xy-coordinate system with origin located at the lower left corner of Ω). We let Q_0 denote the left most room of Ω, i.e., $Q_0 = (0,1)\times(0,1)$. Now consider a function $u \in W^{1,p}(\Omega)$ with $\int_{Q_0} u = 0$. Choose a smooth "bump function" which satisfies: $0 \le \phi(x) \le 1$, $\phi = 1$ on $[1/3, 2/3]$ and $\phi = 0$ off $[0,1]$. Write $u = u_1 + u_2$ where $u_1(x,y) = \phi(x)u(x,y)$.

Since u_1 vanishes off Q_0 and on Q_0 it is dominated in absolute value by $|u|$, we have

$$\int\limits_{\Omega} |u_1|^p \le \int\limits_{Q_0} |u|^p \le M_p^p(Q_0) \int\limits_{Q_0} |\nabla u|^p \lesssim \int\limits_{Q_0} |\nabla u|^p.$$

Now consider u_2. Since this function vanishes for $1/3 \le x \le 2/3$ and hence on each $\Delta_{i,j}$, we have

$$\begin{aligned}
\int\limits_{\Omega} |u_2|^p &= \sum \int\limits_{\Omega_{i,j}} |u_2|^p \\
&\le \sum N^p_{p,\Omega_{i,j}}(\Delta_{i,j}) \int\limits_{\Omega_{i,j}} |\nabla u_2|^p \\
&\le \left(\sup N^p_{p,\Omega_{i,j}}(\Delta_{i,j})\right) \int\limits_{\Omega} |\nabla u_2|^p.
\end{aligned}$$

Also, using the normalization $\int\limits_{Q_0} u = 0$ and the properties of ϕ, we obtain:

$$\begin{aligned}
\int\limits_{\Omega} |\nabla u_2|^p &\lesssim \int\limits_{\Omega} (1-\phi)^p |\nabla u|^p + \int\limits_{\Omega} |u|^p |\phi'|^p \\
&\lesssim \int\limits_{\Omega} |\nabla u|^p + \int\limits_{Q_0} |u|^p \\
&\le (1 + M_p^p(Q_0)) \int\limits_{\Omega} |\nabla u|^p \lesssim \int\limits_{\Omega} |\nabla u|^p.
\end{aligned}$$

Combining these inequalities we obtain, for arbitrary $u \in W^{1,p}(\Omega)$,

$$\int\limits_{\Omega} |u - u_\Omega|^p \lesssim \int\limits_{\Omega} |u - u_{Q_0}|^p \lesssim \sup N^p_{p,\Omega_{i,j}}(\Delta_{i,j}) \int\limits_{\Omega} |\nabla u|^p$$

and hence that $M_p(\Omega) \lesssim \sup N_{p,\Omega_{i,j}}(\Delta_{i,j})$.

Since each $\Omega_{i,j}$ is a b-strip with $b = A_i$, where $A_i = a_1 \cdots a_{i-1} \le 1$, from Theorem 3.5 we may conclude that

$$N^p_{p,\Omega_{i,j}}(\Delta_{i,j}) \lesssim V_{p,\Omega_{i,j}}(h_{i,j}) + A_i^p \frac{4A_i}{A_i^2} \lesssim V_{p,\Omega_{i,j}}(h_{i,j}) + 1.$$

Now the domain $\Omega_{i,j}$ is a rectangle of approximate size $4\times A_i$ with a narrow gap of size A_{i+1} located $1/2^{i-1}$ units to the left of its right-hand boundary. In order to estimate $V_{p,\Omega_{i,j}}(h_{i,j})$, it is easily seen that it suffices to consider only the vertical crosscuts L_{x_i} with abscissa x_i corresponding to this narrow gap. Estimation with the horizontal segment from $h_{i,j}$ to L_{x_i} gives

$$k_{p,\Omega_{i,j}}^{p-1}(h_{i,j},L_{x_i}) \lesssim \left(\frac{4}{A_i^{1/(p-1)}}+\int_{A_{i+1}}^{\infty}\frac{dt}{t^{1/(p-1)}}\right)^{p-1} \lesssim \frac{1}{A_i}+\frac{1}{A_{i+1}^{2-p}}. \tag{4}$$

In addition, $m_2(\Omega_{i,j}(L_{x_i}))=A_i/2^{i-1}$ and hence

$$\begin{aligned} V_{p,\Omega_{i,j}}(h_{i,j}) &\lesssim k_{p,\Omega_{i,j}}^{p-1}(h_{i,j},L_{x_i})\cdot m_2(\Omega_{i,j}(L_{x_i})) \\ &\lesssim \frac{1}{2^{i-1}}+\frac{A_i}{A_{i+1}^{2-p}2^{i-1}} \lesssim 1 \end{aligned}$$

provided

$$A_i = A_{i+1}^{2-p}2^{i-1} \qquad \text{or} \quad a_{i+1}=\frac{A_i^{(p-1)/(2-p)}}{2^{(i-1)/(2-p)}}. \tag{5}$$

We now pass to $\Omega^\star$ and let L_i be the vertical crosscut of $\Omega^\star$ with abscissa x_i. Near $x=x_i$, $\Omega^\star$ is a rectangle of approximate dimensions $2/2^{i-1}\times n_iA_i$ with a narrow gap of size n_iA_{i+1} at $x=x_i$. Assuming (5), we clearly obtain

$$\begin{aligned} k_{p,\Omega^\star}^{p-1}((1/2,0),L_i)\cdot m_2(\Omega^\star(L_i)) &\gtrsim \left(\int_{n_iA_{i+1}}^{n_iA_i}\frac{dt}{t^{1/(p-1)}}\right)^{p-1}\cdot\frac{n_iA_i}{2^{i-1}} \\ &\gtrsim \frac{1}{(n_iA_{i+1})^{2-p}}\cdot\frac{n_iA_i}{2^{i-1}} = n_i^{p-1} \end{aligned}$$

and hence $V_{p,\Omega^\star}=\infty$ provided $\{n_i\}$ is unbounded. But it is straightforward that there exists such a sequence with $\{a_j\}$ satisfying (5). Thus, $M_p(\Omega^\star)=\infty$ by Theorem A and the case $1<p<2$ is complete.

Case 2. $p=2$:

The construction is the same as case 1 except (5) will involve logarithms.

Case 3. $p>2$:

There is a fundamental change in the k_p-metric in this case because $t^{-1/(p-1)}$ is integrable near the origin. This has as a consequence that M_p remains bounded for rectangular domains with a narrow gap no matter how small the gap. Thus, to achieve the same sort of result we need several gaps.

The construction in this case is much the same as in case 1 except we use domains $\Omega'(n,a)$ which are illustrated in Figure 2 above in place of the domains $\Omega(n,a)$.

Note that $\Omega'(n,a)$ is obtained from $\Omega(n,a)$ by adding gaps all of size a and equally spaced apart a distance of $1/n$ in all of the top $n-1$ right rooms. The main point is that no matter how much smaller a is than $1/n$, the k_p metric will essentially ignore these slits in the nonsymmetrized domain Ω. On the other hand, the domain $\Omega^\star$ will appear to have very long narrow corridors since $\delta_{\Omega^\star}$ will be nearly constant (and very small) on intervals of length $1/2^i$.

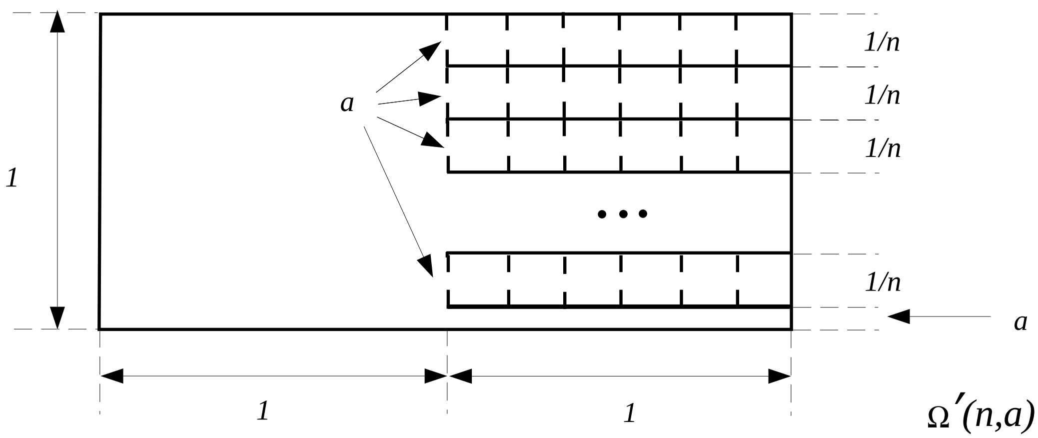

FIGURE 2. The domain $\Omega'(n,a)$ of Example 5.2.

6. THE CASE $p = 1$

In this section we explore to what extent our results about p-Poincaré domains are valid for the case $p = 1$. The geometry of 1-Poincaré domains has been characterized by Maz'ja in his book [Maz85] (see §3.1 and §3.2). For simply connected planar domains, Maz'ja's sufficiency condition has recently been simplified by two of the authors, see [StaSt] and we now described these characterizations.

A crosscut α of Ω separates Ω into two simply connected subdomains: $\Omega_1(\alpha)$ and $\Omega_2(\alpha)$ where $m_2(\Omega_1(\alpha)) \le m_2(\Omega)/2$. We are primarily concerned with three types of crosscuts; namely, general crosscuts α, *segmental crosscuts* σ, i.e., those crosscuts of Ω which are line segments, and hyperbolic geodesic crosscuts τ.

Theorem 6.1. *(Maz'ja) A domain $\Omega \subset \mathbb{C}$ of finite area is a 1-Poincaré domain if and only if*

$$\sup_U \left\{ \frac{m_2(U)}{\Lambda(\partial U \cap \Omega)} \right\} < \infty, \tag{1}$$

where the supremum is taken over all open subsets $U \subset \Omega$ such that $\partial U \cap \Omega$ is a disjoint union of C^∞-curves and $m_2(U) \le m_2(\Omega)/2$. Moreover, the above supremum is comparable to $M_1(\Omega)$.

Theorem 6.2 [StaSt]**.** *Let Ω be a simply connected domain in the plane with finite area. If we let*

$$\mathcal{A} = \sup\left\{\frac{m_2(\Omega_1(\alpha))}{\Lambda(\alpha)} : \alpha \text{ is an arbitrary crosscut of } \Omega\right\},$$
$$\mathcal{G} = \sup\left\{\frac{m_2(\Omega_1(\tau))}{\Lambda(\tau)} : \tau \text{ is a hyperbolic geodesic crosscut of } \Omega\right\}, \text{ and}$$
$$\mathcal{L} = \sup\left\{\frac{m_2(\Omega_1(\sigma))}{\Lambda(\sigma)} : \sigma \text{ is a segmental crosscut of } \Omega\right\},$$

then we have

$$\mathcal{A} \approx \mathcal{G} \approx \mathcal{L} \approx M_1(\Omega). \tag{2}$$

In Example 5.2 we constructed bounded p-Poincaré domains whose Steiner symmetrizations failed to be p-Poincaré domains for each $p > 1$. Interestingly, the restriction on p was necessary as the following stronger version of Theorem C shows.

Theorem 6.3. *If Ω is any connected 1-Poincaré domain whose vertical cross sections have lengths which are uniformly bounded by $b > 0$, then*

$$M_1(\Omega^\star) \lesssim M_1(\Omega) + b.$$

Proof. We let π_1 and π_2 denote the projection operators on the x- and the y-axes respectively. Suppose that Ω is as in the statement of the theorem. Observe that $\Omega^\star$ is simply connected and hence we can define $\mathcal{L}^\star$ to be the quantity in Theorem 6.2 corresponding to segmental crosscuts of the domain $\Omega^\star$. By Theorem 6.2, it suffices to show that $\mathcal{L}^\star \lesssim M_1(\Omega) + b$. To this end, let $\sigma^\star$ be a segmental crosscut of $\Omega^\star$.

Firstly, suppose that the imaginary parts of the endpoints of $\sigma^\star$ are either both positive or both negative or that the real part of one endpoint is either α or β (recall our notation from the definition of centerlines). We then have two possibilities: either $\Omega_1^\star(\sigma^\star)$ is contained in the vertical strip determined by $\sigma^\star$ or else $\Omega_2^\star(\sigma^\star)$ is contained in this strip. In the first case we have $m_2(\Omega_1^\star(\sigma^\star)) \le \Lambda(\sigma^\star)b$. Otherwise,

$$m_2(\Omega_1^\star(\sigma^\star)) \le m_2(\Omega_2^\star(\sigma^\star)) \le \Lambda(\sigma^\star)b.$$

In either case we have $m_2(\Omega_1^\star(\sigma^\star))/\Lambda(\sigma^\star) \le b$.

Henceforth we assume that the imaginary parts of the endpoints of $\sigma^\star$ have different signs and that the real parts are not α or β. Write $\pi_1(\sigma^\star) = [a_1, a_2]$. Without loss of generality, we assume that $\pi_1(\Omega_1^\star(\sigma^\star)) \subset [a_1, \infty)$. Let S be the vertical cross section for Ω with $\pi_1(S) = \{a_2\}$. Clearly, since $\Omega^\star$ is a Steiner symmetric domain, $\Lambda(S) \le 2\Lambda(\sigma^\star)$. If we define $\Omega(S) = \{w \in \Omega : \Re w > a_2\}$ and similarly define $\Omega^\star(S^\star)$ then these sets have the same area and $\Omega_1^\star(\sigma^\star) \setminus \Omega^\star(S^\star)$ is contained in the vertical strip determined by $\sigma^\star$. Consequently,

$$\frac{m_2(\Omega_1^\star(\sigma^\star))}{\Lambda(\sigma^\star)} \le \frac{m_2(\Omega^\star(S^\star))}{\Lambda(\sigma^\star)} + b = \frac{m_2(\Omega(S))}{\Lambda(\sigma^\star)} + b \le 2\frac{m_2(\Omega(S))}{\Lambda(S)} + b.$$

Now observe that $m_2(\Omega(S)) \le m_2(\Omega)/2$ and hence by Theorem 6.1 we have

$$m_2(\Omega(S)) \lesssim M_1(\Omega)\Lambda(S).$$

Combining this with the above yields that

$$\frac{m_2(\Omega_1^\star(\sigma^\star))}{\Lambda(\sigma^\star)} \lesssim M_1(\Omega) + b$$

so that $\mathcal{L}^\star \lesssim M_1(\Omega) + b$ and the proof is complete. □

Now we prove an analog to Theorem A when $p = 1$. Of course, the k_p-metric which played such a prominent role when $p > 1$ is no longer defined. On the other hand, there is a substitute for the quantity $K_{p,\Omega}(w_0)$ which can be used when $p = 1$. Following [SmSt90] we define

$$h_{1,\Omega}(w_1, w_2) = \inf\{\sup_{w\in\gamma} \frac{1}{\delta_\Omega(w)} : \gamma \text{ path in } \Omega \text{ from } w_1 \text{ to } w_2\}$$

and observe that

$$\lim_{p\to 1} \left(\int_\gamma \frac{ds}{\delta_\Omega^{1/(p-1)}} \right)^{p-1} = \sup_{w\in\gamma} \frac{1}{\delta_\Omega(w)}.$$

Hence, it is natural to define

$$K_{1,\Omega}(w_0) = \sup_\tau h_{1,\Omega}(w_0, \tau) \cdot m_2(\Omega(\tau))$$

where the supremum is taken over all hyperbolic geodesics τ with $w_0 \notin \tau$. Similarly, for a b-strip Ω we put

$$V_{1,\Omega}(w_0) = \sup_L h_{1,\Omega}(w_0, L) \cdot m_2(\Omega(L))$$

where the supremum is taken over all vertical crosscuts $L = \Omega_x$ with $x \neq \Re w_0$. Finally, closely related to the above is the quantity

$$\mathcal{V} = \sup \left\{ \frac{m_2(\Omega_1(L))}{\Lambda(L)} : L \text{ is a vertical crosscut of } \Omega \right\}.$$

Lemma 6.4. *Let γ be a hyperbolic geodesic containing the distinct points w_0 and w_1 of Ω. Let τ_1 be a hyperbolic geodesic of Ω which is nearly orthogonal γ at w_1. Then we have*

$$\sup_{w\in\gamma(w_0,w_1)} \frac{1}{\delta_\Omega(w)} \approx h_1(w_0, w_1) \approx h_1(w_0, \tau_1). \tag{3}$$

Proof. If $\rho_\Omega(w_0, w_1) < 1$, then there is nothing to prove since $\delta_\Omega(w) \approx \delta_\Omega(w_0)$ for $w \in \gamma(w_0, w_1)$. Assume that $\rho_\Omega(w_0, w_1) \ge 1$ and let $w^\star$ be the point on $\gamma(w_0, w_1)$ where δ_Ω is a minimum. We modify $w^\star$ slightly so that $\delta_\Omega(w^\star)$ is comparable to the minimum value and $\rho_\Omega(w^\star, w_1) \ge 1$. We can then find a hyperbolic geodesic $\tau^\star$ which is nearly orthogonal to γ at $w^\star$ for which $\Lambda(\tau^\star) \approx \delta_\Omega(w^\star)$.

By Lemma 2.3, $\tau^\star$ separates w_0 from τ_1. Hence any curve from w_0 to τ_1 in Ω must intersect $\tau^\star$ and therefore $h_1(w_0, \tau_1) \gtrsim \delta_\Omega(w^\star)^{-1}$. The rest of the proof is now immediate. □

Theorem 6.5. *Suppose that Ω is a b-strip and γ is a centerline for Ω with $w_0 \in \gamma$. Then,*

(i) $V_{1,\Omega}(w_0) \lesssim K_{1,\Omega}(w_0) + b$,
(ii) $\mathcal{V} \le V_{1,\Omega}(w_0)/2$ *and*
(iii) $M_1(\Omega) \lesssim \mathcal{V} + b$.

Proof. To prove (i), let L be a vertical crosscut with $w_0 \notin L$. We will assume that L is to the right of w_0. Let w_1 be the first intersection of γ with L, starting at w_0. As before, let $w^\star$ be a point on $\gamma(w_0, w_1)$ where δ_Ω is a minimum and $\tau^\star$ a nearly orthogonal hyperbolic geodesic with length comparable to $\delta_\Omega(w^\star)$. By definition and Lemma 6.4

$$h_{1,\Omega}(w_0, L) \lesssim \frac{1}{\delta_\Omega(w^\star)} \approx h_{1,\Omega}(w_0, \tau^\star)$$

and as argued earlier in the proof of Theorem 2.7, $m_2(\Omega(L)) \lesssim m_2(\Omega(\tau^\star)) + b\delta_\Omega(w^\star)$. Combining these inequalities we obtain that

$$h_{1,\Omega}(w_0, L) \cdot m_2(\Omega(L)) \lesssim h_{1,\Omega}(w_0, \tau^\star) \cdot m_2(\Omega(\tau^\star)) + b \le K_{1,\Omega}(w_0) + b$$

which proves (i).

We again let L be a vertical crosscut with $w_0 \notin L$. By definition, either $\Omega_1(L) = \Omega(L)$ or else $m_2(\Omega_1(L)) \le m_2(\Omega(L))$. It is obvious that $h_{1,\Omega}(w_0, L) \ge 2/\Lambda(L)$ and hence

$$\frac{m_2(\Omega_1(L))}{\Lambda(L)} \le \frac{1}{2} h_{1,\Omega}(w_0, L) \cdot m_2(\Omega(L)) \le \frac{1}{2} V_{1,\Omega}(w_0)$$

which proves (ii).

Finally, an argument similar to the proof of Theorem 6.3 shows that $\mathcal{L} \lesssim \mathcal{V} + b$. Hence, (iii) follows from this and Theorem 6.2. □

Theorem 6.6. *If Ω is a simply connected domain with finite area and $w_0 \in \Omega$, then*

$$M_1(\Omega) \lesssim K_{1,\Omega}(w_0) \lesssim \frac{m_2(\Omega)}{\delta_\Omega(w_0)^2} M_1(\Omega).$$

Proof. Let τ be a hyperbolic geodesic in Ω with $w_0 \notin \tau$. It is immediate from the definitions that $m_2(\Omega_1(\tau)) \le m_2(\Omega(\tau))$ and that $\Lambda(\tau)^{-1} \le h_{1,\Omega}(w_0, \tau)$. Multiplying these inequalities and taking suprema yields that $\mathcal{G} \le K_{1,\Omega}(w_0)$ and so by Theorem 6.2 we have $M_1(\Omega) \lesssim K_{1,\Omega}(w_0)$.

To prove the second inequality, we begin again with a hyperbolic geodesic τ in Ω with $w_0 \notin \tau$. Let γ be the hyperbolic geodesic containing w_0 which is orthogonal to τ and let w_τ be the point of intersection of these curves. Suppose first that $h_{1,\Omega}(w_0, w_\tau) \le m_0/\delta_\Omega(w_0)$, where $m_0 \approx 1$ will be specified below. Then inequality (4.2) yields that

$$h_{1,\Omega}(w_0, \tau) \cdot m_2(\Omega(\tau)) \lesssim \frac{m_2(\Omega)}{\delta_\Omega(w_0)} \lesssim \frac{m_2(\Omega)}{\delta_\Omega(w_0)^2} M_1(\Omega), \tag{4}$$

which is the required inequality.

Next suppose that $h_{1,\Omega}(w_0, w_\tau) > m_0/\delta_\Omega(w_0)$. We assume that m_0 was chosen so that Lemma 2.2 (b) now implies that $\rho_\Omega(w_0, w_\tau) > 1$. By Lemma 6.4 (and Lemma 2.2 again) there is a point $w^\star \in \gamma(w_0, w_\tau)$ with the property that $\rho_\Omega(w^\star, w_\tau) \geq 1$ and $h_{1,\Omega}(w_0, \tau) \approx \delta_\Omega(w^\star)^{-1}$. Let $\tau^\star$ be a hyperbolic geodesic in Ω which is nearly orthogonal to γ at $w^\star$ and has length comparable to $\delta_\Omega(w^\star)$. By Lemma 2.3 and the above we have

$$h_{1,\Omega}(w_0, \tau) \cdot m_2(\Omega(\tau)) \lesssim \frac{m_2(\Omega(\tau))}{\delta_\Omega(w^\star)} \lesssim \frac{m_2(\Omega(\tau^\star))}{\Lambda(\tau^\star)}. \tag{5}$$

Now if $m_2(\Omega(\tau^\star)) \leq m_2(\Omega)/2$, then $\Omega_1(\tau^\star) = \Omega(\tau^\star)$ and so by Theorem 6.1 and (5) we obtain

$$h_{1,\Omega}(w_0, \tau) \cdot m_2(\Omega(\tau)) \lesssim M_1(\Omega).$$

On the other hand, if this is not the case, then $w_0 \in \Omega_1(\tau^\star)$. Since $h_{1,\Omega}(w_0, w_\tau) > m_0/\delta_\Omega(w_0)$, we have that $\delta_\Omega(w^\star)$ is small compared to $\delta_\Omega(w_0)$. Thus $\Omega_1(\tau^\star)$ contains a disk of radius comparable to $\delta_\Omega(w_0)$ (centered at w_0) and hence again by Theorem 6.1 we have

$$h_{1,\Omega}(w_0, \tau) \cdot m_2(\Omega(\tau)) \lesssim \frac{m_2(\Omega)}{\delta_\Omega(w^\star)} \lesssim \frac{m_2(\Omega)}{\delta_\Omega(w_0)^2} \frac{m_2(\Omega_1(\tau^\star))}{\Lambda(\tau^\star)} \lesssim \frac{m_2(\Omega)}{\delta_\Omega(w_0)^2} M_1(\Omega)$$

which proves the theorem. □

It is a fact that the class of p-Poincaré domains increases with p; see the Corollary in §4.2 in [Maz85]. In particular, 1-Poincaré domains are p-Poincaré domains for all p, $1 \leq p < \infty$. Surprisingly, there is a fairly general class of domains for which the opposite relation is true.

Theorem 6.7. *Let Ω be a b-strip containing the origin and which projects to the interval (α, β). Suppose that $\Lambda(\Omega_y) \lesssim \Lambda(\Omega_x)$ whenever $0 \leq x < y < \beta$ or $\alpha < y < x \leq 0$ and $|x - y| \leq 1$. If $M_p(\Omega) < \infty$ for some $1 < p < \infty$, then $M_1(\Omega) < \infty$.*

Proof. Assume that $M_p(\Omega) < \infty$ for some p, $1 < p < \infty$. Hence $V_{p,\Omega}(0) < \infty$, by Theorem A. We will show that $\mathcal{V}$ is finite, and then the theorem will follow from Theorem 6.5.

Fix a vertical crosscut L corresponding to an x_L which we assume without loss of generality is positive. First consider the case that $\beta \leq x_L + 1$. Then, by the hypothesis on the cross sectional lengths of Ω, $m_2(\Omega(L)) \lesssim \Lambda(L)$, so that the ratio used to define $\mathcal{V}$ is bounded be a constant comparable to 1.

Next we suppose that $y = x_L + 1 < \beta$, and denote by L_y the vertical crosscut corresponding to y. Then any curve connecting L to L_y has arc length at least 1, and $\delta_\Omega(w) \lesssim \Lambda(L)$ for $w \in \Omega$ with $x \leq \Re w \leq y$ by our assumptions on Ω. Thus

$$\frac{1}{\Lambda(L)} \lesssim k_p^{p-1}(L, L_y) \leq k_p^{p-1}(0, L_y)\,,$$

and so

$$\frac{m_2(\Omega_1(L))}{\Lambda(L)} \leq \frac{m_2(\Omega(L))}{\Lambda(L)} \lesssim \frac{m_2(\Omega(L_y)) + \Lambda(L)}{\Lambda(L)} \lesssim k_p^{p-1}(0, L_y) \cdot m_2(\Omega(L_y)) + 1\,.$$

This last term, by definition, is bounded by $V_{p,\Omega}(0) + 1$, so $\mathcal{V} \lesssim V_{p,\Omega}(0) + 1 < \infty$ and the proof is complete.

The next example shows that the approximate monotone decrease in cross sectional length of Ω was a necessary assumption in Theorem 6.7.

Example 6.8. Suppose that $\alpha \geq 1$. Let $x_0 = 1$ and $x_n = x_{n-1} - x_{n-1}^\alpha/2$ for $n = 1, 2, \ldots$. Then, $x_n \to 0$ and we define

$$\Omega = (0,2) \times (-1,1) \setminus \bigcup_{n=1}^{\infty} \{(x,y) \mid x = x_n \text{ and } x_n^\alpha \leq |y| < 1\}.$$

Then $M_p(\Omega) < \infty$ if and only if $p \geq \alpha$.

Since Ω is a 2-strip, we may use Theorem A to see this. Let $w_0 = (1,0)$ and note that $\delta_\Omega((x,0)) \approx x^\alpha$ for $0 < x < 1$. Suppose $1 < p < \alpha + 1$ and fix x, $0 < x < 1/2$. Let $L = \Omega_x$, so that $m_2(\Omega(L)) = x$ and

$$k_p^{p-1}(w_0, L) \approx \left(\int_x^1 \frac{dt}{t^{\alpha/(p-1)}} \right)^{p-1} \approx x^{p-1-\alpha}.$$

Thus $k_p^{p-1}(w_0, L) \cdot m_2(\Omega(L)) \approx x^{p-\alpha}$, so $V_{p,\Omega}(w_0) = \infty$ and $M_p(\Omega) = \infty$ if $1 < p < \alpha$, while $V_{p,\Omega}(w_0) < \infty$ and $M_p(\Omega) < \infty$ if $\alpha \leq p < \alpha + 1$. For $p = 1 < \alpha$, consideration of the crosscuts $L_n = \Omega_{x_n}$ shows that $\mathcal{V} = \infty$ and so $M_1(\Omega) = \infty$ by Theorems 6.5 and 6.6, while $\mathcal{V}$ and $M_1(\Omega)$ are finite if $p = 1 = \alpha$. The computation for $p \geq \alpha + 1$ is similar and will be omitted.

Example 6.9. Let $\Omega = \{(x,y) \mid 0 < y < e^{-|x|}, -\infty < x < \infty\}$. Then, for $1 \leq p < \infty$, $M_p(\Omega) < \infty$.

As noted above the class of p-Poincaré domains increases with p, and so it is sufficient to prove $M_1(\Omega) < \infty$. Let $w_0 = (0, 1/2)$, and note that Ω is a 1-strip. Thus, by Theorem 6.5, it is enough to show that $\mathcal{V} < \infty$. To see this let $x > 0$ and $L = \Omega_x$. Then $\Lambda(L) = e^{-x} = \int_x^\infty e^{-t}\, dt = m_2(\Omega_1(L))$, and so $\mathcal{V} = 1$ and the proof is complete.

Remark. In Example 4.10 it was shown that a p-Poincaré domain can not contain arbitrarily long *rectangular passages*. The current example shows, however, that a p-Poincaré domain can contain arbitrarily long *rectangles*. We also remark that the demonstration that $M_p(\Omega) < \infty$ in this example could have been based on Lemma 3.1 and Theorem 3.5.

We end this section with an example which shows the necessity of the boundedness assumption in Theorem 6.3. The example is of a 1-Poincaré domain whose Steiner symmetrization fails to be a p-Poincaré domain for all $p \geq 1$.

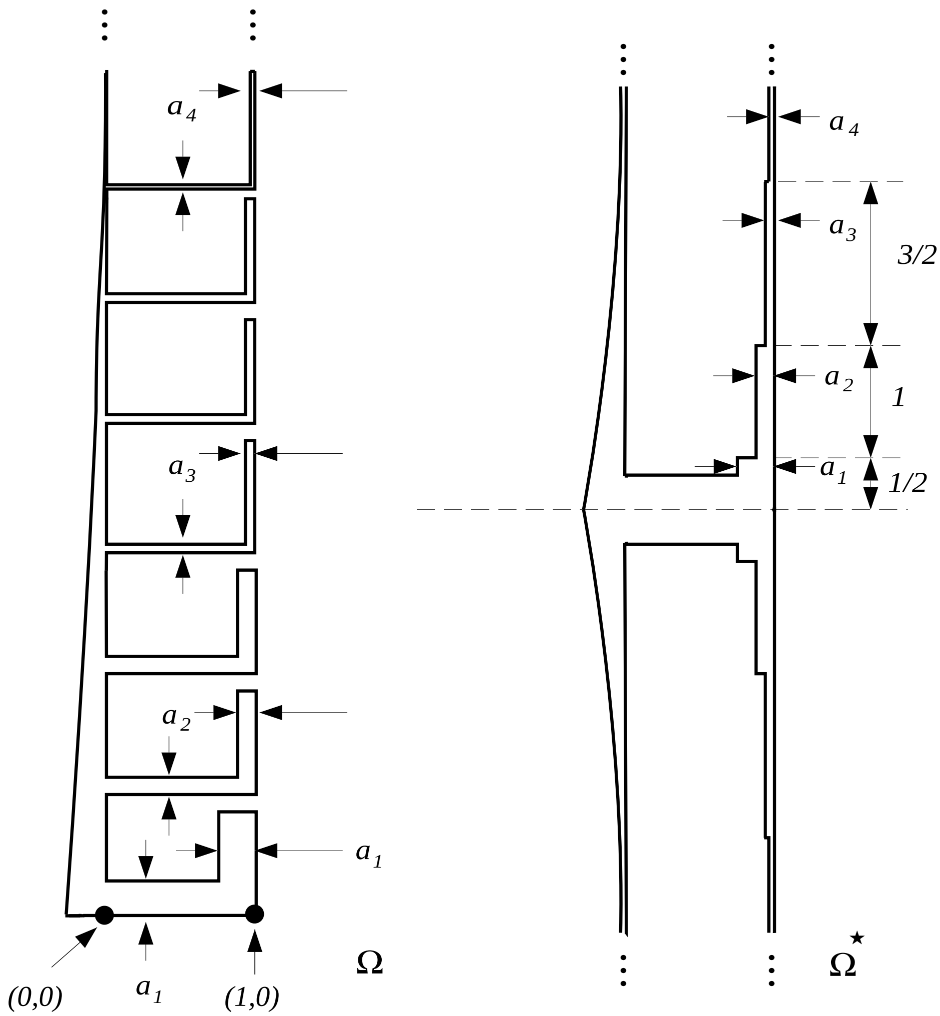

FIGURE 3. This domain Ω is a p-Poincaré domain for all $p \geq 1$, but its Steiner symmetrization $\Omega^\star$ is not a p-Poincaré domain for any $p \geq 1$.

Example 6.10. The domain Ω is illustrated in Figure 3, where the x- and y- axes are determined by the location of the two points $(0,0)$ and $(1,0)$. The left most boundary curve is given by $-x = 2^{-y}$, $\quad y \geq 0$, and the parameters a_i shall be specified presently.

Note that corresponding to each a_i, Ω will have i "elbows" of height and width equaling one, and with thickness a_i. To make the p-Poincaré inequalities fail on $\Omega^\star$ for all p will require no restrictions on the parameters. Since $\Omega^\star$ contains rectangular passages of arbitrarily long lengths, this follows from Example 4.10 when $p > 1$ and it is an immediate consequence of Theorem 6.1 for $p = 1$. Alternatively, this can be seen directly by a simple computation using, for a given such (vertical) passage P, the piecewise linear test function which is given by $u(x, y) = y$ on P and is constant on the two remaining complementary components to show that such a domain cannot be a p-Poincaré domain for any $p \geq 1$.

To make Ω a 1-Poincaré domain, we first of all must require that $\sum na_n < \infty$, to ensure that $|\Omega| < \infty$. By Theorem 6.6, it suffices to show that $L < \infty$. Due to the geometry of Ω, the only segmental crosscuts σ which need to be considered are the horizontal crosscuts σ_i which protrude to the left of the $y-$axis at the lower left bases of the first elbows associated with a_i for $i \geq 2$. For sufficiently large i, $\Omega(\sigma_i)$ consists of all the elbows associated with a_j for $j \geq i$ together with the part of Ω to the left of the $y-$axis with $y > 1 + 2 + \cdots + (i-1) \equiv y_i$. Hence,

$$m_2(\Omega_1(\sigma_i)) \approx \sum_{j \geq i} ja_j + \int_{y_i}^{\infty} 2^{-t}\, dt \lesssim \sum_{j \geq i} ja_j + 2^{-y_i},$$

and $\Lambda(\sigma_i) = 2^{-y_i}$. Thus if the a_i satisfy

$$\sum_{j \geq i} ja_j \lesssim 2^{-i^2/2},$$

for all i, then Ω will be a 1-Poincaré domain. For example one can take $a_i = 2^{-i^2}/i$.

7. BLD Images of b-Strips

In this section we extend our results on b-strips to domains that are bilipschitzian images of them. The geometric definition of a b-strip is unstable under small bilipschitzian perturbations while, on the other hand, we observed in §4 that the bilipschitzian image a p-Poincaré domain will continue to be one. Thus it should be expected that our main results extend to a broader class of domains. This extension will be at the expense of the simple geometry of b-strips; it no longer will be as routine to verify that a given domain satisfies the geometric hypotheses.

We say that a homeomorphism $T : \Omega' \to \Omega$ is a locally uniformly bilipschitz mapping if, for any point in Ω', the restriction of T to a suitably small neighborhood of that point is a bilipschitz mapping with constants independent of the point. In this case we write $\Omega \approx \Omega'$. It is clear that an equivalent definition is that $\Lambda(T(\eta)) \approx \Lambda(\eta)$, for all rectifiable curves $\eta \subset \Omega'$. For this reason such a mapping T is also called a bounded length distortion, or BLD mapping, see [Väi] and also [Geh]. Observe that a BLD image of b-strip can be quite general. For example, an unbounded b-strip can be mapped by a BLD homeomorphism to a domain spiraling out to infinity.

We next collect some simple observations concerning these mappings.

Lemma 7.1. *Suppose that $\Omega = T(\Omega')$, where T is a BLD mapping. Then*

(a) $\delta_\Omega(T(w')) \approx \delta_{\Omega'}(w')$, $w' \in \Omega'$;
(b) $k_{p,\Omega}(T(w_1'), T(w_2')) \approx k_{p,\Omega'}(w_1', w_2')$, $w_1', w_2' \in \Omega'$;
(c) $m_2(E) \approx m_2(T(E))$, $E \subset \Omega'$ *a Borel set;*
(d) $M_p(\Omega) \approx M_p(\Omega')$, $1 \leq p < \infty$.

Proof. Part (a) is an obvious consequence of the definition of a BLD map, and (b) follows immediately from (a). The Jacobian of a BLD mapping T is comparable to 1, and so (c) is clear. Since $|\nabla(u \circ T)| \approx |\nabla u|$ almost everywhere, for $u \in C^1(\Omega')$, (d) is a consequence of the change of variables formula for Sobolev functions (see §2.2 of [Zie]).

Since the definition of $K_{p,\Omega'}(w_0')$ involves the hyperbolic geometry of Ω' and this geometry is not preserved by a BLD mapping T, it is not clear how this quantity compares to $K_{p,\Omega}(T(w_0'))$ when $\Omega \approx \Omega'$.

Theorem 7.2. *Let Ω and Ω' be simply connected planar domains with finite area. Suppose that $\Omega = T(\Omega')$, where T is a BLD mapping, and let $w_0' \in \Omega'$. Then, for $1 \le p < \infty$, $K_{p,\Omega'}(w_0') \approx K_{p,\Omega}(T(w_0'))$.*

Proof. Suppose first that $1 < p < \infty$. Let $w_0 = T(w_0')$, $\delta_0 = \delta_\Omega(w_0)$ and $\delta_0' = \delta_{\Omega'}(w_0')$. Since $\Omega \approx \Omega'$ is an equivalence relation, it is only necessary to prove that $K_{p,\Omega}(w_0) \lesssim K_{p,\Omega'}(w_0')$. We start by establishing a basic lower bound for K_p.

By using the geometry of hyperbolic geodesics in the unit disk and conformal invariance, we see that there is a fixed integer N and a collection of hyperbolic geodesics $\tau_1', \dots, \tau_N'$ in Ω' such that $\rho_{\Omega'}(w_0', \tau_i') = 1$, for $1 \le i \le N$, and

$$\{w' \in \Omega' \mid \rho_{\Omega'}(w_0', w') \ge 2\} \subset \bigcup_{i=1}^{N} \Omega'(\tau_i')\,.$$

It follows that $m_2(\Omega') \lesssim m_2(\cup\,\Omega'(\tau_i'))$ and, for $1 \le i \le N$, $k_p^{p-1}(w_0', \tau_i') \approx (\delta_0')^{p-2}$. Thus, we obtain the inequality

$$(\delta_0')^{p-2} m_2(\Omega') \lesssim \sum_{i=1}^{N} (\delta_0')^{p-2} m_2(\Omega'(\tau_i')) \lesssim K_{p,\Omega'}(w_0')\,.$$

Suppose that τ is a hyperbolic geodesic in Ω with $0 < \rho_\Omega(w_0, \tau) \le m_0$. Let γ be the hyperbolic geodesic in Ω containing w_0 which is orthogonal to τ at a point w_τ. Lemma 2.2 implies that there is a $\lambda > 1$ for which

$$\Lambda(\gamma(w_0, w_\tau)) \lesssim \lambda^{m_0} \delta_0 \quad \text{and} \quad \delta_\Omega(w) \gtrsim \lambda^{-m_0} \delta_0$$

for all $w \in \gamma(w_0, w_\tau)$. Hence, $k_p^{p-1}(w_0, \tau) \cdot m_2(\Omega(\tau)) \lesssim \lambda^{m_0 p} \delta_0^{p-2} m_2(\Omega)$. Combining this with parts (a) and (c) of Lemma 7.1 and the above, we have

$$\begin{aligned} k_p^{p-1}(w_0, \tau) \cdot m_2(\Omega(\tau)) &\lesssim \lambda^{m_0 p} \delta_0^{p-2} m_2(\Omega) \\ &\approx \lambda^{m_0 p} (\delta_0')^{p-2} m_2(\Omega') \lesssim \lambda^{m_0 p} K_{p,\Omega'}(w_0')\,. \end{aligned}$$

This reduces the proof to the consideration of hyperbolic geodesics τ in Ω with $\rho_\Omega(w_0, \tau) \ge m_0$, where m_0 is large.

Now let $\tau = \tau_4$ be a hyperbolic geodesic in Ω with $\rho_\Omega(w_0, \tau_4) \ge m_0$, where $m_0 \ge 4$ will be specified below. We must show that $k_p^{p-1}(w_0, \tau_4) \cdot m_2(\Omega(\tau_4)) \lesssim K_{p,\Omega'}(w_0')$.

Again, let γ be the hyperbolic geodesic in Ω containing the point w_0 which is orthogonal to τ_4 at a point w_4. Let w_3 and w_2 be the points on $\gamma(w_0, w_4)$ for which

$$\rho_\Omega(w_3, w_4) = 1 \quad \text{and} \quad \rho_\Omega(w_2, w_4) = 2.$$

By Lemma 2.3, there are disjoint hyperbolic geodesics τ_2, τ_3 which are nearly orthogonal to γ at w_2, w_3, satisfy $\Lambda(\tau_i) \approx \delta_\Omega(w_i)$ and are also disjoint from τ_4. Thus, $\Omega(\tau_4) \subset \Omega(\tau_3) \subset \Omega(\tau_2)$ and $w_0 \notin \Omega(\tau_2)$. By Lemma 2.6 we have

$$k_p^{p-1}(w_0, \tau_4) \cdot m_2(\Omega(\tau_4)) \lesssim k_p^{p-1}(w_0, \tau_3) \cdot m_2(\Omega(\tau_3))$$

and we focus our attention on τ_3.

Now consider the images of these curves under the mapping T^{-1}. We label the image curves γ', τ_2', τ_3' and the points w_0', w_2', w_3'. We caution the reader that these curves in Ω' are *not* hyperbolic geodesics but they are crosscuts and they do partition Ω' into six disjoint subregions. Moreover, by Lemma 7.1 the lengths of these curves, the areas that they determine and the distances $\delta_{\Omega'}$ are all comparable to the corresponding lengths, areas and distances in Ω.

Denote by σ' the hyperbolic geodesic in Ω' containing w_0' and w_3', and define w_3'' if necessary so that $\sigma'(w_0', w_3'')$ intersects τ_3' just at w_3''. Define $w_1' \in \sigma'(w_0', w_3'')$ by the equation

$$\left(\int\limits_{\sigma'(w_0',w_1')} \frac{ds}{\delta_{\Omega'}^{1/(p-1)}}\right)^{p-1} = \epsilon \left(\int\limits_{\sigma'(w_1',w_3'')} \frac{ds}{\delta_{\Omega'}^{1/(p-1)}}\right)^{p-1}, \tag{1}$$

where $1/2 > \epsilon > 0$ will be specified below. Let τ_1' be a hyperbolic geodesic of Ω' which is nearly orthogonal to σ' at the point w_1' and whose length is comparable to $\delta_{\Omega'}(w_1')$.

Suppose that $\Omega'(\tau_3') \subset \Omega'(\tau_1')$. Then, by the definition of w_1', Lemma 7.1 and Lemma 2.6 we see that

$$\begin{aligned}
k_{p,\Omega}^{p-1}(w_0, \tau_3) \cdot m_2(\Omega(\tau_3)) &\approx k_{p,\Omega'}^{p-1}(w_0', \tau_3') \cdot m_2(\Omega'(\tau_3')) \\
&\lesssim \left(\int\limits_{\sigma'(w_0',w_3'')} \frac{ds}{\delta_{\Omega'}^{1/(p-1)}}\right)^{p-1} \cdot m_2(\Omega'(\tau_1')) \\
&= (1 + \frac{1}{\epsilon^{1/(p-1)}})^{p-1} \left(\int\limits_{\sigma'(w_0',w_1')} \frac{ds}{\delta_{\Omega'}^{1/(p-1)}}\right)^{p-1} \cdot m_2(\Omega'(\tau_1')) \\
&\approx \frac{1}{\epsilon} k_{p,\Omega'}^{p-1}(w_0', \tau_1') \cdot m_2(\Omega'(\tau_1')) \leq \frac{1}{\epsilon} K_{p,\Omega'}(w_0')
\end{aligned}$$

which proves the theorem. Thus, it suffices to prove that there is an ϵ sufficiently small and m_0 sufficiently large so that $\tau_1' \cap \tau_3' = \emptyset$.

Suppose now that $\tau_1' \cap \tau_3' \neq \emptyset$. We first show that

$$\delta_{\Omega'}(w') \lesssim \Lambda(\tau_3') \tag{2}$$

for all $w' \in \sigma'(w_1', w_3'')$. Let w' be such a point. By Lemma 2.2, we may as well assume that $\rho_{\Omega'}(w_1', w') \geq 1$. Let τ' be the hyperbolic geodesic through w' which is nearly orthogonal to σ' and has length comparable to $\delta_{\Omega'}(w')$. By Lemma 2.3, τ' is disjoint from τ_1'. By assumption, there is a point w_1'' where the curve τ_1' first intersects τ_3' starting at w_1'. We get a Jordan region using the curves $\tau_1'(w_1', w_1'')$, $\sigma'(w_1', w_3'')$ and $\tau_3'(w_1'', w_3'')$. It follows from the Jordan curve theorem that τ' must intersect τ_3'. In other words, the curve τ_3' intersects *both* hyperbolic geodesics τ_1', τ' and hence (2) follows from Lemma 2.5.

Next we show the opposite inequality

$$\delta_{\Omega'}(w') \approx \Lambda(\tau') \gtrsim \Lambda(\tau_3') \tag{3}$$

for this same point by essentially the same argument. By Lemma 7.1 and Lemma 2.5 we know that any curve which intersects both τ_2' and τ_3' must have length greater than a constant multiple of $\delta_\Omega(w_3) \approx \Lambda(\tau_3')$. Thus, if τ' has this property then (3) follows. In particular, this must be the case if $w' \notin \Omega'(\tau_2')$ since we already know that $\tau' \cap \tau_3' \neq \emptyset$.

Finally, suppose that $w' \in \Omega'(\tau_2')$ and $\tau' \cap \tau_2' = \emptyset$. By our construction of the curve σ' we know that $w' \notin \Omega'(\tau_3')$. Moreover, $w_0' \notin \Omega'(\tau_2')$ and hence there is point $w_2'' \in \sigma'$ with the property that $\sigma'(w_2'', w') \cap \tau_2' = \{w_2''\}$. Observe next that $\sigma'(w', w_3') \cap \tau_2' = \emptyset$. For otherwise there would be a $w_2''' \in \tau_2$ where $\sigma'(w', w_3')$ first crosses τ_2'. Then, the Jordan region in Ω' bounded by $\sigma'(w_2'', w_2''')$ and $\tau_2'(w_2'', w_2''')$ would enclose part of τ', thus preventing it from tending to a boundary point, which is impossible.

Now let $z' \in \gamma'(w_2', w_3')$ be the first intersection of this curve, starting at w_2', with $\sigma'(w_2'', w_3')$. Since w_3' belongs to both curves, such a point exists. Again we obtain a Jordan region in Ω'; bounded by $\tau_2'(w_2', w_2'')$, $\sigma'(w_2'', z')$ and $\gamma'(w_2', z')$, which encloses part of τ'. Since τ' must tend to the boundary of Ω' there must be a point $z'' \in \tau' \cap \gamma'(w_2', z')$. By our construction, the distance to the boundary of each point on $\gamma'(w_2', w_3')$ is comparable to $\delta_\Omega(w_3)$. Thus, $\Lambda(\tau') \gtrsim \delta_{\Omega'}(w_3')$ and (3) is proved.

By (2) and (3) we have that $\delta_{\Omega'}(w') \approx \Lambda(\tau_3')$ for all $w' \in \sigma'(w_1', w_3'')$. In addition, the Gehring-Hayman theorem yields that $\Lambda(\sigma'(w_1', w_3'')) \lesssim \Lambda(\tau_1') + \Lambda(\tau_3') \approx \Lambda(\tau_3')$ since $\tau_1' \cap \tau_3' \neq \emptyset$. It follows that the quasihyperbolic distance in Ω' between w_1' and w_3'' satisfies $k_{2,\Omega'}(w_1', w_3'') \lesssim 1$ and similarly with the hyperbolic metric.

Hence, by our assumption and Lemma 7.1 we obtain from the triangle inequality that

$$\begin{aligned} m_0 \leq \rho_\Omega(w_0, \tau_3) &\lesssim \rho_{\Omega'}(w_0', w_3'') \\ &\leq \rho_{\Omega'}(w_0', w_1') + \rho_{\Omega'}(w_1', w_3'') \lesssim \rho_{\Omega'}(w_0', w_1') + 1 \end{aligned}$$

so that $\rho_{\Omega'}(w_0', w_1') \gtrsim m_0$. This last inequality requires that m_0 be sufficiently large and we chose m_0 to satisfy this requirement. This insures that $\Lambda(\sigma'(w_0', w_1')) \gtrsim$

$\delta_{\Omega'}(w_1')$ and thus by (1), (2), (3) we have

$$\begin{aligned}\frac{\delta_{\Omega'}(w_1')^{p-1}}{\delta_{\Omega'}(w_1')} &\lesssim \left(\int\limits_{\sigma'(w_0',w_1')} \frac{ds}{\delta_{\Omega'}^{1/(p-1)}}\right)^{p-1} \\ &= \epsilon \left(\int\limits_{\sigma'(w_1',w_3'')} \frac{ds}{\delta_{\Omega'}^{1/(p-1)}}\right)^{p-1} \lesssim \epsilon \frac{\delta_{\Omega'}(w_1')^{p-1}}{\delta_{\Omega'}(w_1')},\end{aligned}$$

which is clearly impossible for ϵ sufficiently small. This completes the proof in the case $p > 1$.

The case $p = 1$ is proved by modifying the above argument as was done in §6 to prove Theorem 6.5. In particular, occurrences of k_p^{p-1} should be replaced by h_1. The only major change is that the two cases to consider are: $h_{1,\Omega}(w_0,\tau) \leq m_0\delta_0^{-1}$ or the opposite inequality. In the latter case we replace equation (1) with $h_{1,\Omega'}(w_0',w_1') = \epsilon h_{1,\Omega'}(w_1',w_3'')$, where $\epsilon = m_0^{-1}$. $\square$

The following result is an immediate consequence of Theorem A in the introduction and Theorem 7.2. (Recall that more is true for $p = 1$; see Theorem 6.6.)

Corollary 7.3. *Suppose $\Omega \approx \Omega'$, where Ω' is a b-strip, and let $w_0 \in \Omega$. Then, for $1 < p < \infty$, $M_p(\Omega) < \infty$ if and only if $K_{p,\Omega}(w_0) < \infty$.*

The results in this paper should be compared to those in [EvHar]. In that paper Evans and Harris introduced generalized ridged domains and studied the Poincaré inequality on such domains. In their work a generalized ridge plays a role analogous to that of the centerline γ in the proof of Theorem 3.5. We wish to emphasize, however, that not all b-strips are generalized ridged domains. It can be shown, for example, that the domain in Example 6.8 is not a generalized ridge domain, while it clearly is a b-strip. On the other hand, generalized ridged domains are closely related to b-strips. In particular, examples 6.1, 6.2 and 6.3 of [EvHar] are all b-strips, and it is readily verified that their condition for the validity of the Poincaré inequality on those domains is equivalent to the condition that $K_{p,\Omega}(w_0) < \infty$. It seems likely that any generalized ridge domain is, in fact, a BLD image of a b-strip.

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Dated: 2/28/1994

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