THE GEOMETRY OF POINCARÉ DISKS

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Abstract. A simply connected planar domain Ω of finite area is said to be a Poincaré disk if there exists a finite positive constant K such that

$$\int_\Omega |u|\,dx \le K \int_\Omega |\nabla u|\,dx$$

for all functions u which are C^1 on Ω and integrate to zero. In this paper we establish geometric necessary and sufficient conditions for Ω to be a Poincaré disk. Our criteria, which are reminiscent of the isoperimetric inequality and simplify a characterization of Maz'ja, state that the smaller area determined from a crosscut of Ω must be bounded by a constant multiple of the length of the crosscut. We show that this characterization is valid for three different types of crosscuts: line segments, hyperbolic geodesics, and general crosscuts. We also obtain a characterization of those conformal mappings which map the disk onto a Poincaré disk in terms of an integral growth condition. We use techniques from geometric function theory and hyperbolic geometry.

1. INTRODUCTION

Let Ω be a domain (i.e., an open connected set) in $\mathbb{C}$ with finite area, which we denote by $|\Omega|$. For a number p, $1 \le p < \infty$, we let $W^{1,p}(\Omega)$ denote the first order Sobolev space of those (real-valued) L^p-integrable functions u on Ω whose distributional partial derivatives $D^\alpha u$, $|\alpha| = 1$, also lie in $L^p(\Omega)$. The Sobolev space $W^{1,p}(\Omega)$ becomes a Banach space when endowed with the norm

$$\|u\|_{W^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \sum_{|\alpha|=1} \|D^\alpha u\|_{L^p(\Omega)}.$$

For convenience, we let ∇u denote the distributional gradient vector of first order partials and write $\|\nabla u\|_{L^p(\Omega)}$ for $\sum_{|\alpha|=1} \|D^\alpha u\|_{L^p(\Omega)}$. There are several good general references on the Sobolev spaces and their functions, we cite: [Zie], [Ada], [Maz-85], and Chapter 7 of [GT]. For an integrable function h on Ω we let h_Ω denote its average value, i.e.,

$$h_\Omega = \frac{1}{|\Omega|} \int_\Omega h(x)\ dx.$$

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Definition. The domain Ω is said to be a *p-Poincaré domain* if there is a finite constant K such that the *p-Poincaré inequality*:

$$\|u - u_\Omega\|_{L^p(\Omega)} \leq K\|\nabla u\|_{L^p(\Omega)} \tag{1}$$

is valid for all $u \in W^{1,p}(\Omega)$. The *best p-Poincaré constant* is the smallest number $K_p(\Omega)$ for which the above inequality holds.

If $p = 1$ and Ω is a simply connected domain, then we say that Ω is a *Poincaré disk.*

Remark. By the fundamental density result in [MS] the functions in $C^1(\Omega)$ are dense in $W^{1,p}(\Omega)$ and hence to determine whether or not (1) holds it suffices to test the inequality with smooth functions.

The Poincaré inequalities are useful tools for problems which involve partial differential equations. It is known, for example, that Ω is a 2-Poincaré domain precisely when the the so called *Neumann problem* of finding a function $u \in W^{1,2}(\Omega)$ satisfying the (distributional) PDE $-\Delta u + \lambda u = f$ for a given scalar $\lambda > 0$ and function $f \in L^2(\Omega)$ always has a solution (see, e.g., [Maz-68]). It is thus important to have criteria which guarantee that a domain satisfy a Poincaré inequality, and in case it does, to give some sort of estimates on the corresponding best contant. In the present paper we restrict attention to the case in which $p = 1$, and Ω is a simply connected planar region (of finite area). We will give a complete geometric characterization of these domains which satisfy the 1-Poincaré inequality, and furthermore we shall give three different geometric quantities which are always comparable to $K_1(\Omega)$.

We employ the following notation. We let $\mathbb{D}$ denote the unit disk $\{|z| < 1\}$ in the complex plane. For a point z_0 in the plane and a positive number r, we let $B(z_0, r)$ denote the disk $\{z : |z - z_0| < r\}$. The symbols "ℓ", "d", and "diam" will be used to denote, respectively, Euclidean arclength, distance, and diameter. The area measure on $\mathbb{R}^2$ will be written as $dA(z)$, while $|E|$ will denote the area of a (measurable) set $E \subset \mathbb{R}^2$. We use the following convention for comparing different positive constants c_1 and c_2: the relation "$c_1 \lesssim c_2$" shall mean that c_1 is bounded above by some absolute constant times c_2. When c_1 and c_2 are comparable, i.e., when $c_1 \lesssim c_2 \lesssim c_1$ we shall write $c_1 \approx c_2$. It is in general true that the classes of p-Poincaré domains increase with with p. This result can be deduced from the Corollary in §4.2 in [Maz-85] using Lemma 5 of [SS-90], however, a nice direct proof of this fact will appear (as the proof of Theorem 1.8) in the forthcoming book [HKM]. In particular, 1-Poincaré domains are p-Poincaré domains for all p, $1 \leq p < \infty$.

We henceforth assume that Ω is a simply connected domain in the plane with finite area. A *crosscut* of Ω is an open non self-intersecting arc in Ω such that $\overline{\alpha} \setminus \alpha$ consists of one or two points of $\partial\Omega$. Thus a crosscut in Ω separates Ω into two simply connected subdomains: $\Omega_1(\alpha)$ and $\Omega_2(\alpha)$. For convenience of notation, we shall always assume that these two components are labeled so that

$$|\Omega_1(\alpha)| \leq |\Omega_2(\alpha)|.$$

We make use of three types of crosscuts. Apart from the general crosscuts, we also work with *segmental crosscuts* σ, i.e., those crosscuts of Ω which are line segments. The third class of crosscuts we consider are those which are *hyperbolic geodesics* for Ω. Recall that a crosscut γ is a hyperbolic geodesic for Ω, if it is a geodesic for the hyperbolic metric on Ω. This is equivalent to γ being the image under a Riemann mapping $g:\mathbb{D}\to\Omega$ of a diameter of D. We refer to [Ahl] and to [Bea] for more details on hyperbolic geometry. We are ready to state the main results of this paper.

In [Maz-85](see §3.1 and §3.2) Maz'ja gave a general characterization of 1-Poincaré domains which is valid in $\mathbb{R}^n$. His proof used the co-area formula from geometric measure theory and in two dimensions says the following:

Theorem. *(Maz'ja) A domain $\Omega\subset\mathbb{C}$ of finite area is a 1-Poincaré domain if and only if*

$$\sup_{\mathcal{G}}\left\{\frac{|\mathcal{G}|}{\ell(\partial\mathcal{G}\cap\Omega)}\right\}<\infty,$$

where the supremum is taken over all open subsets $\mathcal{G}\subset\Omega$ such that $\partial\mathcal{G}\cap\Omega$ is a disjoint union of C^∞-curves and $|\mathcal{G}|\le(1/2)|\Omega|$. Moreover, the above supremum is comparable to $K_1(\Omega)$.

Motivated by this result, we use techniques of geometric function theory to give a new proof of Maz'ja's Theorem with a much simplier geometric hypothesis:

Theorem A. *Let Ω be a simply connected domain in the plane with finite area $|\Omega|$. If we let*

$$A=\sup\left\{\frac{|\Omega_1(\alpha)|}{\ell(\alpha)}:\alpha \text{ is an arbitrary crosscut of } \Omega\right\},$$
$$G=\sup\left\{\frac{|\Omega_1(\gamma)|}{\ell(\gamma)}:\gamma \text{ is a hyperbolic geodesic crosscut of } \Omega\right\},\ \text{and}$$
$$L=\sup\left\{\frac{|\Omega_1(\sigma)|}{\ell(\sigma)}:\sigma \text{ is a segmental crosscut of } \Omega\right\},$$

then we have

$$A\approx G\approx L\approx K_1(\Omega). \tag{2}$$

Theorem A is proved in Sections 3 and 4. Section 3 is devoted to establishing the comparablility of A and L. That A and $K_1(\Omega)$ are comparable can be deduced from Maz'ja's Theorem above; instead, we shall give a simpler and more geometric proof of the relation $A\approx K_1(\Omega)$ relying on hyperbolic geometry and Whitney decompositons. These ideas will be used again in the final Section 5 in which we shall obtain the following characterization of those conformal mappings on $\mathbb{D}$ which map onto Poincaré disks.

Theorem B. *Let Ω be as in Theorem A, and let $g:\mathbb{D}\to\Omega$ be a Riemann map. Then Ω is a Poincaré disk if and only if there is a constant $C(g)$ depending only on g such that the inequality*

$$\int_{C_z}|g'(\zeta)|^2\,dA(\zeta)\le C(g)|g'(z)|\left(1-|z|^2\right), \tag{3}$$

holds for all $z \in \mathbb{D}$, $z \neq 0$. Here C_z denotes the smaller complementary subdomain of $\mathbb{D}$ determined by the hyperbolic geodesic which passes through z and is tangent to the circle $\{\zeta : |\zeta| = |z|\}$. Moreover, if we let $C = \inf C(g)$, where the g ranges over all such Riemann mappings, then C is comparable to each of the constants in (3).

2. Preliminaries

In this section we outline the tools we need to prove our theorems. We start with the hyperbolic metric on the unit disk $\mathbb{D}$. This can be defined by (see [Ahl, p. 2] or see [Bea])

$$\begin{aligned}\rho_{\mathbb{D}}(z_1, z_2) &= \inf\left\{\int_\gamma \frac{2|dz|}{1-|z|^2} : \gamma \text{ is an arc in } \mathbb{D} \text{ from } z_1 \text{ to } z_2\right\}\\ &= \log\frac{|1-\overline{z_1}z_2| + |z_1 - z_2|}{|1-\overline{z_1}z_2| - |z_1 - z_2|}.\end{aligned}$$

This distance is invariant under conformal self-mappings of the disk or put another way, Möbius transformations of the disk are isometries in the hyperbolic metric. The geodesics in this metric are the hyperbolic lines which are just the circular arcs which are perpendicular to $\partial\mathbb{D}$ (including the diameters). We will need the following geometric fact, see §7.22-7.23 in Beardon's book [Bea]. Note the contradistinction to Euclidean geometry, there is a *unique* hyperbolic line which meets two disjoint hyperbolic lines orthogonally.

Lemma 2.1. *The shortest distance between disjoint hyperbolic lines is the distance measured along a uniquely determined mutually orthogonal line.*

The invariance of the hyperbolic metric under conformal self-mappings of the disk results in a natural conformally invariant metric on any simply connected proper subset $\Omega \subset \mathbb{C}$. If $g : \mathbb{D} \to \Omega$ is any conformal map, the *hyperbolic distance* on Ω is given by $\rho_\Omega(w_1, w_2) = \rho_{\mathbb{D}}(z_1, z_2)$, where $w_i = g(z_i)$ for $i = 1, 2$. The function

$$h_\Omega(w) = \frac{2}{|g'(z)|(1-|z|^2)}, \qquad \text{where } w = g(z), \tag{4}$$

satisfies

$$h_\Omega(w)|dw| = \frac{2|dz|}{1-|z|^2}, \qquad w = g(z).$$

The hyperbolic geodesics for Ω are just the images under g of geodesics for $\mathbb{D}$. The next result is a symmetric and conformally invariant version of Theorem 10.8 on page 311 in [Pomm].

Theorem 2.2(Pommerenke). *Let $g : \mathbb{D} \to \Omega$ be a conformal mapping. If $a \in \mathbb{D}$, and $\epsilon > 0$ is given, then there exists a set $E = E(g, \epsilon, a) \subset \partial\mathbb{D}$ of harmonic measure $\omega_a(E) < \epsilon$ with the following property. If $\gamma[a, e^{i\theta}]$ denotes the hyperbolic geodesic of $\mathbb{D}$ determined by a and $e^{i\theta}$ then for each $e^{i\theta} \in \partial\mathbb{D} \setminus E$, we have*

$$\ell(f(\gamma[a, e^{i\theta}])) < c(\epsilon)\delta_\Omega(g(a)),$$

where, as indicated, the constant $c(\epsilon)$ depends only on ϵ.

The metric ρ_Ω can be computed by integrating $h_\Omega(w)$ over arcs in Ω. However, $h_\Omega(w)$ is not explicitly computable in terms of Ω alone. A useful substitute is the *quasi-hyperbolic metric* on Ω, introduced by Gehring and Palka [GP]. For a domain $\Omega \subsetneqq \mathbb{R}^n$ and $x \in \Omega$, let $\delta_\Omega(x)$ denote the Euclidean distance from x to the boundary of Ω. The quasi-hyperbolic distance from x_1 to x_2 in Ω is defined to be

$$k_\Omega(x_1, x_2) = \inf \left\{ \int_\gamma \frac{ds}{\delta_\Omega(x)} : \gamma \text{ is an arc in } \Omega \text{ from } x_1 \text{ to } x_2 \right\}.$$

Here ds denotes integration with respect to arclength.

The quasi-hyperbolic metric is closely related to the hyperbolic metric. Indeed, if Ω is a simply connected domain in the complex plane, then it follows from the fundamental Koebe Distortion Theorem (see Corollary 1.4 on page 22 of [Pomm]) that

$$\frac{1}{2} \le h_\Omega(w)\delta_\Omega(w) \le 2. \tag{5}$$

It follows that

$$\frac{1}{2}\rho_\Omega \le k_\Omega \le 2\rho_\Omega\,. \tag{6}$$

Due to the geometric nature of its definition, k_Ω is thus very useful in obtaining estimates for the hyperbolic metric. One particularly useful estimate is the following inequality which is in fact true for arbitrary proper domains in $\mathbb{R}^n$ (see Lemma 2.1 in [GP]).

Theorem 2.3(Gehring-Palka). *For any pair of points $a, b \in \Omega$, we have*

$$\left| \log \frac{\delta_\Omega(a)}{\delta_\Omega(x)} \right| \le k_\Omega(a, b) \qquad (a, b \in \Omega).$$

Another consequence that we will need is the following result which essentially appears on the bottom of page 21 in [Pomm].

Lemma 2.4. *Let $g : \mathbb{D} \to \Omega$ be a conformal mapping. Then,*

$$e^{-2\rho_{\mathbb{D}}(z_1,z_2)} \le \frac{(1-|z_1|^2)|g'(z_1)|}{(1-|z_2|^2)|g'(z_2)|} \le e^{2\rho_{\mathbb{D}}(z_1,z_2)}$$

An alternate approach to the quasihyperbolic metric can be based on the Whitney decompostion of Ω. Let $\mathcal{W}$ be a Whitney decomposition of the domain $\Omega \subset \mathbb{C}$ into closed dyadic squares with disjoint interiors. This means that the coordinates of the vertices of each square are dyadic rational numbers and that the diameter of each square $Q \in \mathcal{W}$, $\operatorname{diam}(Q)$, is comparable to its distance to $\partial\Omega$. See Chapter 6 of Stein's book [Ste] for the existence and basic properties of such a decomposition. The quasi-hyperbolic metric k_Ω on Ω can be shown to be equivalent to minimum number of Whitney cubes connecting two points. We will not require that result but we do need to observe that Whitney cubes are approximately quasihyperbolic balls of radius one.

Lemma 2.5. *Let Ω be a simply connected subset of the plane and $\mathcal{W}$ a Whitney decomposition. There exist positive constants c_1, c_2 such that for any Whitney cube $Q \in \mathcal{W}$ with center x_Q, if B_1, B_2 are the hyperbolic balls of Ω, centered at z_Q, of radii c_1, c_2 then $B_1 \subset Q \subset B_2$.*

Finally, we will need the following two separation theorems from plane topology. For a proof of the first we refer to Theorem VI.7.1 in [New], and for the second to Theorem 1.9 on page 31 in [Pomm].

Lemma 2.6. *If the points a and b of a simply connected domain D in the (extended) plane are separated in D by the closed set F in the (extended) plane then they are separated in D by a component of $F \cap D$.*

Theorem 2.7(Janiszewski). *Let A_1 and A_2 be two closed sets in the plane such that $A_1 \cap A_2$ is connected. If the points a and b are neither separated by A_1 nor A_2 then they are not separated by $A_1 \cup A_2$.*

3. Comparability of the constants A and L

The comparability of these constants is split up into several lemmas. By homogeneity of these constants we may assume throughout that $|\Omega| = 1$. The first lemma is a consequence of the classical isoperimetric inequality which states that of all Jordan domains having a given perimeter, a disk has the maximum possible area.

Lemma 3.1. *Let α be a crosscut of Ω having endpoints a_1 and a_2 (possibly the same) and having length $\ell(\alpha) < 1$. If $|\Omega| = 1$ and if the open line segment S with endpoints a_1 and a_2 has empty intersection with Ω, then $|\Omega_1(\alpha)|/\ell(\alpha) \leq 1/\pi$.*

Proof. Assume that $a_1 \neq a_2$. The curves α and S combine to form a Jordan curve with an interior domain D. We assert that $\Omega_1(\alpha) \subset D$. To prove this, we consider an arbitrary point $x \in \alpha$. Choose $\epsilon = \epsilon(x) > 0$ such that $B(x, \epsilon) \subset \Omega$ and then choose a point $y \in D \cap B(x, \epsilon)$. Let Ω' denote the component of $\Omega \setminus \alpha$ determined by y. Observe that $\Omega' \subset D$. Indeed, if this were not the case then there would be a point $y' \in \Omega' \setminus D$ and a path η in Ω' from y to y'. By the Jordan Curve Theorem, this path η must cross ∂D, and hence S, however, $S \cap \Omega = \emptyset$ which implies that η must cross $\partial\Omega$ which is a contradiction. Invoking the classical isoperimetric inequality on D, we deduce that

$$|\Omega'| \leq |D| \leq \frac{(\ell(\partial D))^2}{4\pi} < \frac{\ell(\alpha)^2}{\pi} < \frac{1}{2} = \frac{|\Omega|}{2}.$$

This gives that $\Omega' = \Omega_1$, and by the same token we obtain

$$\frac{|\Omega_1(\alpha)|}{\ell(\alpha)} =\leq \frac{\ell(\alpha)}{\pi} \leq \frac{1}{\pi}.$$

This takes care of the case that $a_1 \neq a_2$. In case $a_1 = a_2$, the result follows immmediately from the isoperimetric inequality.

Lemma 3.2. *If Ω is a planar domain with area $|\Omega| = 1$, then the constant L of Theorem A is at least $\frac{1}{20}$.*

Proof. We define

$$A(t) = |\Omega \cap \{x > t\}|$$

and

$$L(t) = \ell(\Omega \cap \{x = t\}).$$

Translating if necessary, we may assume that $A(0) = \frac{1}{2}$. We may furthermore assume that $A(t)/L(t) < \frac{1}{20}$ for $0 \leq t \leq \frac{1}{5}$. Observe that by Fubini's theorem, we have $A(t) = \int_t^\infty L(s)\,ds$. Therefore either $A(\frac{1}{5}) = 0$ or else

$$20 < \frac{L(t)}{A(t)} = -\frac{d}{dt}(\log A(t)),$$

for almost every t on the interval $0 < t < 1/5$. In the latter case, we have

$$20\left(\frac{1}{5}\right) < \int_0^{1/5} -\frac{d}{dt}(\log A(t))\,dt = \log \frac{1}{2A(\frac{1}{5})},$$

so that $A(\frac{1}{5}) < e^{-4}/2$. Using a symmetric argument on negative values of t, we conclude that in any case we may assume that

$$\left|\Omega \setminus \{|x| < \frac{1}{5}\}\right| < e^{-4}.$$

A symmetric argument in the y-variable allows us to assume that

$$|\Omega \setminus \{|y| < \frac{1}{5}\}| < e^{-4}.$$

Consequently we have that $|\Omega \setminus \{|x| < \frac{1}{5}$ and $|y| < \frac{1}{5}\}| < 2e^{-4}$. On the other hand $|\Omega| = 1$ and $|\{|x| < \frac{1}{5}$ and $|y| < \frac{1}{5}\}| = \frac{4}{25}$ from which we obtain $1 < \frac{4}{25} + 2e^{-4} < 1$ which is a contradiction.

Theorem 3.3. $A \approx L$.

Proof. Since $L \leq A$ we must prove the opposite inequality with a constant. Our basic assumption that $|\Omega| = 1$ and Lemma 3.2 show that it suffices to assume that we are given a crosscut α of Ω with $\ell(\alpha) < \frac{1}{4}$ and to construct a segmental crosscut σ with

$$\frac{|\Omega_1(\alpha)|}{\ell(\alpha)} \lesssim \frac{|\Omega_1(\sigma)|}{\ell(\sigma)}.$$

Moreover, by Lemma 3.1, we may assume that the segment S between the endpoints of α has a nonempty intersection with Ω. We write

$$S \cap \Omega = \bigcup_j \sigma_j,$$

where the σ_j's are segmental crosscuts.

We enclose α in the interior of a disk D with radius less than $\ell(\alpha)$ (and hence area less than 1/4). The components of $\Omega \setminus \overline{D}$ separate into two types: the B_k's which are disjoint from $\Omega_1(\alpha)$, and the C_l's which are subsets of $\Omega_1(\alpha)$. See Figure 1.

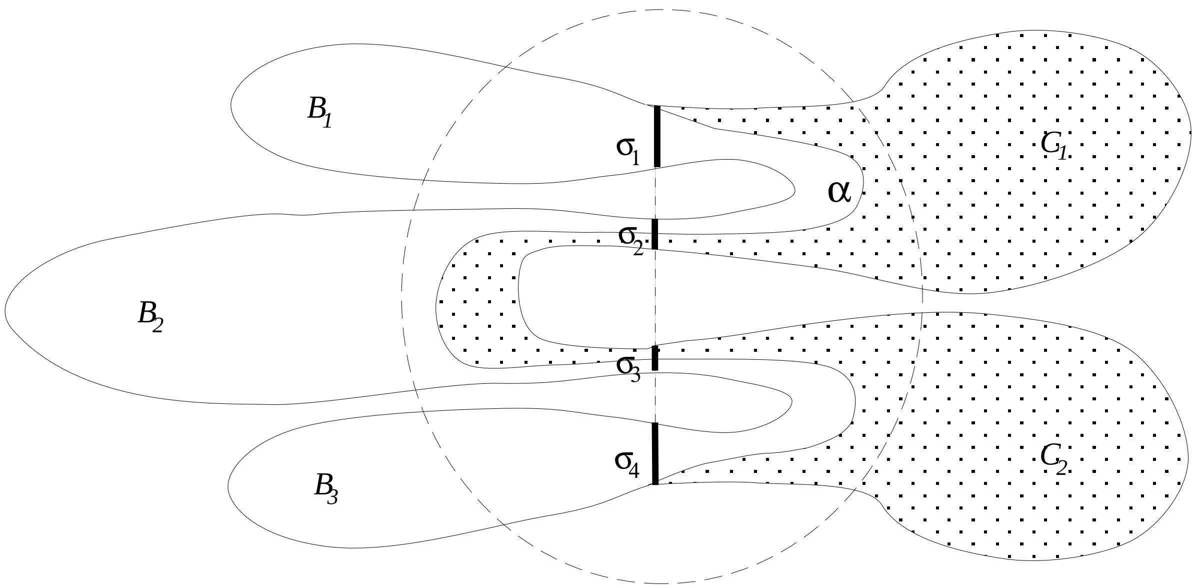

FIGURE 1. The shaded region is $\Omega_1(\alpha)$.

It will suffice to prove the following inequality.

Claim. $|\cup_j \Omega_1(\sigma_j)| \geq \frac{1}{2}|\cup C_l|$.

Indeed, the claim implies that (the first inequality is straightforward)

$$\sup \frac{|\Omega_1(\sigma_j)|}{\ell(\sigma_j)} \geq \frac{\sum |\Omega_1(\sigma_j)|}{\sum \ell(\sigma_j)} \geq \frac{|\bigcup \Omega_1(\sigma_j)|}{\ell(S)} \geq \frac{|\bigcup C_l|}{2\ell(\alpha)}.$$

On the other hand, if $|\Omega_1(\alpha)| > 5|D|$, then

$$|\bigcup C_l| = |\Omega_1(\alpha) \setminus D| \geq |\Omega_1(\alpha)| - |D| \geq \frac{4}{5}|\Omega_1(\alpha)|$$

and hence

$$\frac{|\Omega_1(\alpha)|}{\ell(\alpha)} \leq \frac{5}{2} \sup \frac{|\Omega_1(\sigma_j)|}{\ell(\sigma_j)} \lesssim L.$$

In case $|\Omega_1(\alpha)| \leq 5|D|$, we have

$$\frac{|\Omega_1(\alpha)|}{\ell(\alpha)} \leq \frac{5|D|}{\ell(\alpha)} \leq \frac{5\pi\ell(\alpha)^2}{\ell(\alpha)} \leq \frac{5\pi}{4} \lesssim L$$

by Lemma 3.2.

This shows that proving the above claim will suffice. To this end, we assume that $|\cup\Omega_1(\sigma_j)| < \frac{1}{2}|\cup C_l|$. Since each σ_j is a crosscut lying in D, it follows that each B_k and each C_l is either contained in some $\Omega_1(\sigma_j)$ or is disjoint from all of them. By the preceding inequality it is obvious that there must be a component C_0 which is disjoint from $\cup\Omega_1(\sigma_j)$. Furthermore, there must be a component B_0 which is also disjoint from this union for otherwise $|\cup B_k| < 1/4$ which would imply that $1/2 \leq |\Omega_2(\alpha)| < 1/4 + |D| \leq 1/2$ which is a contradiction.

It now follows from Lemma 2.6, that B_0 and C_0 are not separated in Ω by the entire segment S. However, the special geometry of our situation yields the following direct proof: Suppose that B_0 and C_0 are separated in Ω by S. Let $b_0 \in B_0$, $c_0 \in C_0$ and γ be a curve in Ω from b_0 to c_0. There exists an $\epsilon > 0$ such that $\delta_\Omega(z) > \epsilon$ for z on the curve γ and hence each segment σ_i crossed by γ must have length at least 2ϵ. Hence $\gamma \cap S$ is contained in a *finite* union: $\sigma_{n_1} \cup \cdots \cup \sigma_{n_k}$. Assuming that $\gamma \cap \sigma_{n_1}$ is nonempty, let s_1 be the first crossing of γ and s_2 the last. Now consider the curve $\tilde{\gamma}$ obtained by following γ to s_1, next following the segment $[s_1, s_2]$ and then following γ from s_2 to c_0. By our assumption, $B_0 \cup C_0 \subset \Omega_2(\sigma_{n_1})$ and hence b_0 and c_0 lie on the same side of $\Omega \setminus \sigma_{n_1}$. It follows that $\tilde{\gamma}$ can be modified slightly so as to *not* intersect σ_{n_1}. By induction, we conclude that there is a curve from b_0 to c_0 in Ω which is disjoint from S. This contradicts the separation assumption and we conclude that B_0 and C_0 are not separated in Ω by S.

This can be reformulated as saying that b_0 and c_0 are not separated (in the extended plane) by the closed set $A_1 = S \cup \partial\Omega$. Since the closed set $A_2 = S \cup \alpha$ is contained in the disk D, it also cannot separate b_0 from c_0. But since the intersection $A_1 \cap A_2 = S$ is certainly connected, it follows from Janiszewski's Theorem 2.7 that the set $A_1 \cup A_2 = S \cup \alpha \cup \partial\Omega$ does not separate b and c which is false because $\alpha \cup \partial\Omega$ already separates b and c.

We thank the referee for simplifying the final separation argument (using Janiszewski's theorem) in the above proof. We also note that Janiszewski's theorem could also have been used on the other separation result above.

4. Hyperbolic Geometry and Shadows of Whitney Squares

For a particular point $w \in \Omega$ and Whitney cube Q, we define the *shadow* $S_w(Q)$ to be the union of all Whitney squares Q_1 which contain at least one point w' such that the hyperbolic geodesic joining w to w' intersects Q. Thus, assuming light travels along hyperbolic geodesics, the shadow of a Whitney square Q is the union of all Whitney squares which contain "dark spots" from a light source stationed at w due to the obtacle Q. Smith and the second author showed (see Theorem 8 in [SS-90]) the following general relationship between shadows and Poincaré constants.

Theorem 4.1. *For a simply connected planar domain Ω of finite area and points $w \in \Omega$, we let*

$$S(w) = \sup_{Q \in \mathcal{W}} \frac{|S_w(Q)|}{\text{diam } (Q)}, \quad \text{and} \quad S = \inf_{w \in \Omega} S(w).$$

Then

$$K_1 \lesssim S.$$

In order to complete the proof of Theorem A, it will suffice to establish the following comparability chain:

$$A \lesssim K_1 \lesssim S \lesssim G \lesssim A.$$

The second inequality is Theorem 4.1, the last inequality is obvious, and the first inequality can easily be verified by using the distance function δ_Ω to construct, for each crosscut γ of Ω, a corresponding "collar" function whose Rayleigh-Ritz quotient is comparable to $|\Omega_1(\gamma)|/\ell(\gamma)$. For the details of a more general result we refer to Lemma 3.2.2 on page 165 of [Maz-85]. It is the verification of the third inequality to which the rest of this section is devoted.

Theorem 4.2. *For a simply connected domain Ω in the plane of area $|\Omega| < \infty$, we have*

$$S \approx G.$$

As noted above, we need only show that the left-hand side is dominated by the right-hand side. The strategy for the proof of Theorem 4.2 will be to find an "appropriate" point $w_0 \in \Omega$ for which $S(w_0) \lesssim G$. The next two results will show a natural way to select such an appropriate point.

Lemma 4.3. *Suppose that $f : \mathbb{D} \to \mathbb{C}$ is analytic with finite Dirichlet integral $\int_{\mathbb{D}} |f'(z)|^2\, dA(z)$. If we let $\varphi_t(z)$ denote the conformal automorphism of $\mathbb{D}$ defined by*

$$\varphi_t(z) = \frac{z+t}{1+tz} \qquad (-1 < t < 1), \text{ and}$$

$$F(t) = \int_{\mathbb{D}\cap\{\Re z>0\}} |(f\circ\varphi_t)'(z)|^2\, dA(z) = \int_{\varphi_t(\mathbb{D}\cap\{\Re z>0\})} |f'(z)|^2\, dA(z),$$

then F is continuous on $[-1,1]$, $F(-1) = \int_{\mathbb{D}} |f'(z)|^2\, dA(z)$, and $F(1) = 0$.

Proof. The result follows from absolute continuity of the measure $\mu(E) = \int_E |f'(z)|^2\, dA(z)$ on $\mathbb{D}$.

Corollary 4.4. *If $f : \mathbb{D} \to \mathbb{C}$ is analytic with $\int_{\mathbb{D}} |f'(z)|^2\, dA(z) = 1$ then there exists a conformal automorphism φ of $\mathbb{D}$ such that*

$$\int_{\mathbb{D}\cap\{\Re z>0\}} |(f\circ\varphi)'(z)|^2\, dA(z) = \frac{1}{2},$$

and

$$\int_{\mathbb{D}\cap\{\Im z>0\}} |(f\circ\varphi)'(z)|^2\, dA(z) = \frac{1}{2}.$$

Proof. Apply Lemma 4.3 to get a $t_1 \in (-1,1)$ such that $\int_{\mathbb{D}\cap\{\Re z>0\}} |(f\circ\varphi_{t_1})'|^2\, dA = \frac{1}{2}$. Next, reapply Lemma 4.3, this time to the function $g(z) = f\circ\varphi_{t_1}(iz)$ to obtain a $t_2 \in (-1,1)$ such that $\int_{\mathbb{D}\cap\{\Re z>0\}} |(f\circ\varphi_{t_2})'(z)|^2\, dA(z) = \frac{1}{2}$. It is clear that the composition $\varphi = \varphi_{t_1}\circ\varphi_{t_2}$ has the desired properties.

Definition. *We call a Riemann mapping of $\mathbb{D}$ onto Ω balanced if it satisfies the conclusions of the above Corollary. Under such a map, we refer to the image of $0 \in \mathbb{D}$ as a center point of Ω.*

Theorem 4.5. *If Ω is a simply connected planar domain with unit area and w_0 is a center point of Ω, then $G \gtrsim 1/\delta_\Omega(w_0)$.*

Proof. Let $f : \mathbb{D} \to \Omega$ be a balanced Riemann map with $f(0) = w_0$. Next, either $\mathbb{D} \cap \{\Re z > 0, \Im z > 0\}$ or $\mathbb{D} \cap \{\Re z < 0, \Im z > 0\}$ gets mapped by f to a set of area at least $\frac{1}{4}$. For definiteness, let us assume that the first of these sets possesses this property. It follows that $\mathbb{D} \cap \{\Re z < 0, \Im z < 0\}$ also must get mapped by f to a set of area at least $\frac{1}{4}$.

By Theorem 2.2, there exists an angle θ_0 with $\pi/2 < \theta_0 < \pi$ such that for an absolute constant c, we have

$$\ell(f(\gamma[0, e^{i\theta_0}])) \leq c\delta_\Omega(w_0).$$

The result now follows since the hyperbolic geodesic $f(\gamma[0, e^{i\theta_0}])$ divides Ω into two parts each of which has area at least $\frac{1}{4}$.

Proof of Theorem 4.2. Again we use Corollary 4.4 to produce a conformal mapping $f\colon D \to \Omega$ which maps the upper and lower as well as the left and right halves of $\mathbb{D}$ onto sets of area $\frac{1}{2}$. We fix a Whitney decomposition $\mathcal{W}$ of Ω and a Whitney square Q_0 which contains the point $w_0 = f(0)$. Next, we separate the Whitney squares into the following two classes:

$$\begin{aligned} \mathcal{W}^1 &= \left\{Q^1 \in \mathcal{W} : \rho_\Omega\left(Q^1, Q_0\right) \leq s\right\}, \text{ and} \\ \mathcal{W}^2 &= \mathcal{W} \setminus \mathcal{W}^1, \end{aligned}$$

where ρ_Ω denotes the hyperbolic metric on Ω and the parameter s is an absolute constant which we shall specify later.

Consider now a square $Q^1 \in \mathcal{W}^1$. Theorem 2.3 in conjuction with the comparability (6) of the hyperbolic metric for Ω with the quasi-hyperbolic metric k_Ω gives that

$$\text{diam } (Q^1) \approx \text{diam } (Q_0).$$

Next, we use this relation along with Theorem 4.5, and the fact that $|\Omega| = 1$, to deduce that

$$\frac{|S_{w_0}(Q^1)|}{\text{diam } (Q^1)} \leq \frac{1}{\text{diam } (Q^1)} \approx \frac{1}{\text{diam } (Q_0)} \approx \frac{1}{\delta_\Omega(w_0)} \lesssim G.$$

We now consider an arbitrary Whitney square $Q^2 \in \mathcal{W}^2$. By Theorem 2.5 there exists an absolute constant d such that

$$\rho_\Omega\text{-diam } (Q) < d,$$

for each $Q \in \mathcal{W}$. Next, we choose a Euclidean radius $r_1, 0 < r_1 < 1$ such that the shorter subarc of $\partial\mathbb{D}$ determined by any hyperbolic geodesic which is disjoint from

$r_1\mathbb{D}$ has length less than $\pi/4$. Let r_0 be another such radius corresponding to the angle $\pi/2$. We can take s to be

$$s = \rho_{\mathbb{D}}(0, r_1) + 6d.$$

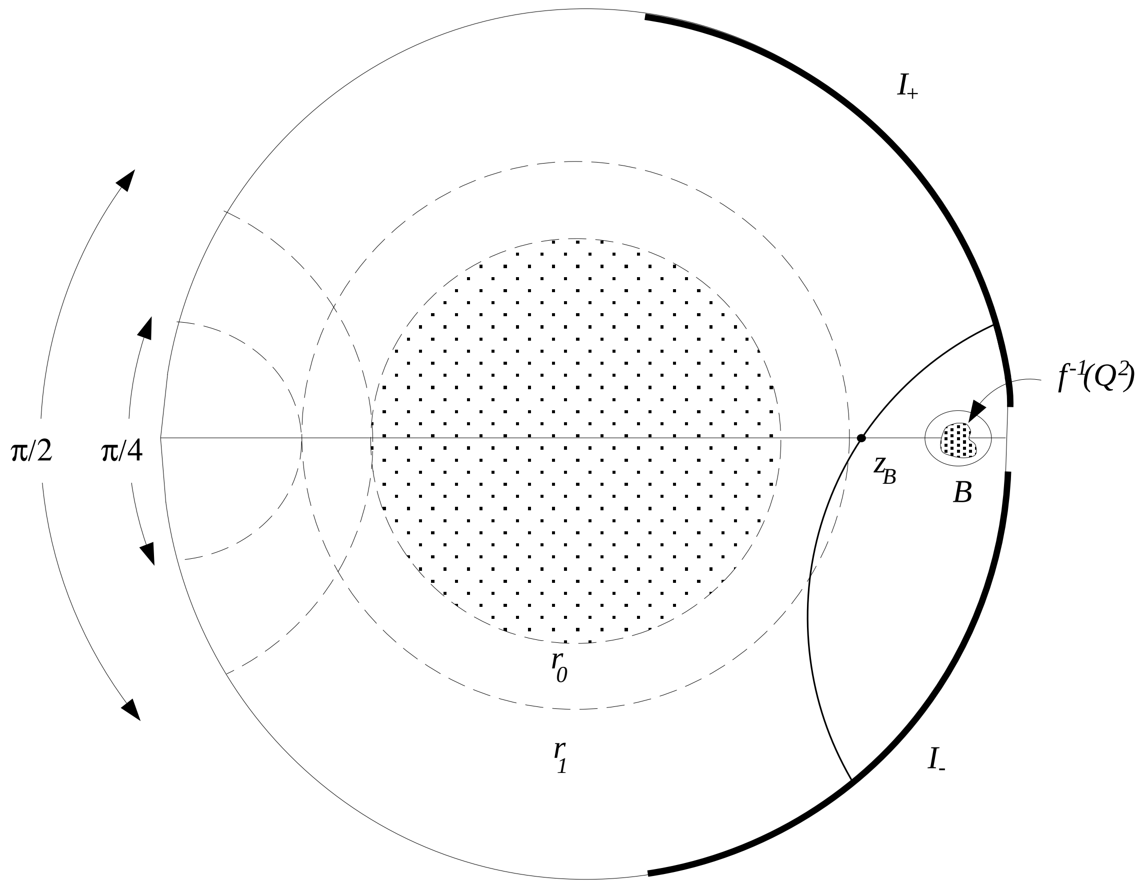

FIGURE 2

Since f, being conformal, is a hyperbolic isometry, we have

$$\rho_{\mathbb{D}}\text{-diam } (f^{-1}(Q^2)) = \rho_{\Omega}\text{-diam } (Q^2).$$

Hence, by definition of $\mathcal{W}^2$, we have

$$\rho_{\mathbb{D}}\,(f^{-1}(Q^2), 0) \geq \rho_{\Omega}\,(Q^2, Q_0) > \rho_{\mathbb{D}}(0, r_1) + 6d.$$

Let B be any hyperbolic disk in $\mathbb{D}$ with hyperbolic radius $2d$ and hyperbolic center any point of $f^{-1}(Q^2)$, so that

$$f^{-1}(Q^2) \subset B, \text{ and}$$

$$\rho_{\mathbb{D}}\,(\partial B, f^{-1}(Q^2)) \geq d.$$

By the triangle inequality we have

$$\rho_{\mathbb{D}}(B,0) \geq \rho_{\mathbb{D}}(0,r_1) + 2d.$$

We let the point z_B be the unique point on the segment between 0 and the center of B which lies a hyperbolic distance $2d$ from B. Consider now the collection of hyperbolic geodesics in $\mathbb{D}$ passing through z_B which are disjoint from both $r_0\mathbb{D}$ and B. The endpoints of these geodesics form a symmetric pair of arcs on $\partial\mathbb{D}$ of ω_{z_B}-harmonic measure which is bounded below by an absolute lower bound, say η_0. By applying Theorem 2.2 with $\epsilon = \eta_0/4$, we conclude that one of the above geodesics, let us call it γ, must have an image geodesic with length less than $c(\eta_0/4)\delta_\Omega(f(z_B))$. If we denote the center of Q^2 by w_{Q^2}, and let $z_{Q^2} = f^{-1}(w_{Q^2})$, then from $\rho_{\mathbb{D}}(z_B, z_{Q^2}) \approx 1$, we get that $\delta_{\mathbb{D}}(z_B) \approx \delta_{\mathbb{D}}(z_{Q^2})$. We may now infer from Lemma 2.4 that

$$|f'(z_B)| \approx |f'(z_{Q^2})|.$$

Since, the distortion inequalities (5) may be rewritten as (see also (4))

$$\delta_\Omega(f(z)) \approx |f'(z)|\delta_{\mathbb{D}}(z) \qquad (z \in \mathbb{D}),$$

it follows that

$$\ell(f(\gamma)) \lesssim \delta_\Omega(f(z_B)) \approx \delta_\Omega(w_{Q^2}).$$

Finally, by our choice of f and since $\gamma \subset \{\Re z > 0\}$, the part of $\mathbb{D}\backslash\gamma$ determined by B gets mapped to a set of area less than $\frac{1}{2}$ so that this must be smaller component. But since this region clearly contains the shadow $S_{w_0}(Q^2)$, we obtain the desired inequality:

$$\frac{|S_{w_0}(Q^2)|}{\text{diam }(Q^2)} \lesssim \frac{|\Omega_1(f(\gamma))|}{\delta_\Omega(w_{Q^2})} \lesssim \frac{|\Omega_1(f(\gamma))|}{\ell(f(\gamma))} \lesssim G.$$

5. Conformal mapping onto Poincaré disks

In this section we make use of the development in Section 4 to prove Theorem B.

Lemma 5.1. *Let $\mathcal{W}$ be a Whitney decomposition of a simply connected planar domain $\Omega \subsetneqq \mathbb{C}$. For any positive number s, there corresponds an integer $n = n(s)$, such that for any fixed $Q' \in \mathcal{W}$, the number of Whitney squares $Q \in \mathcal{W}$ with $\rho_\Omega(Q, Q') \leq s$ is no greater than n.*

Proof. We say that two Whitney squares are neighbors if they meet (even at a vertex). Since adjacent Whitney squares have comparable sidelenths, it follows that any Whitney square can have at most n_0 neighbors. The positive integer n_0 can be computed explicitly, but there is no need to do so. For a finite subset $\mathcal{F} \subset \mathcal{W}$, we define the "star"-operation on $\mathcal{F}$ as follows:

$$\mathcal{F}^\star = \mathcal{F} \cup \{Q \in \mathcal{W} : Q \text{ is a neighbor of some } Q' \in \mathcal{F}\}.$$

By their defining properties, it follows that any Whitney square has hyperbolic diameter comparable to 1. Consequently,

$$\rho_\Omega(\cup\mathcal{F}, \partial[\cup\mathcal{F}^\star]) \geq c_0,$$

for some absolute positve constant c_0. This latter fact implies that if we successively apply the star-operation to the collection $\mathcal{F}_0 = \{Q'\}$ at least s/c_0 times, then the resulting collection of Whitney squares will contain all $Q \in \mathcal{W}$ with $\rho_\Omega(Q, Q') \le s$. The previous fact can be used to calculate fixed upper bounds for the number of Whitney squares in each of these successive collections. The proof of Lemma 5.1 is therefore complete.

Proof of Theorem B. By homogeneity we may assume that $|\Omega| = 1$. For a given Riemann map $g : \mathbb{D} \to \Omega$, we let $C(g)$ denote the smallest positive constant (possibly infinity) for which inequality (4) is valid. To see that $G \lesssim C(g)$, we begin with a hyperbolic geodesic γ of Ω, and let $\widetilde{\gamma}$ denote the hyperbolic geodesic $g^{-1}(\gamma)$ of $\mathbb{D}$. Letting z denote the point of $\widetilde{\gamma}$ which is closest to the origin, we certainly have

$$|g(C_z)| \ge |\Omega_1(\gamma)|.$$

On the other hand, the distortion inequalities (5) yield

$$|g'(z)|(1 - |z|^2) \approx \delta_\Omega(g(z)) \lesssim \ell(\gamma).$$

By considering the appropriate ratios and taking suprema over all such geodesics γ, we get $G \lesssim C(g)$, and hence,

$$G \lesssim C.$$

We must also obtain an upper bound for C in terms of one of the five comparable constants A, G, L, $K_1(\Omega)$, and S. Let $f : \mathbb{D} \to \Omega$ denote a balanced Riemann map and fix a Whitney decomposition $\mathcal{W}$ of Ω with $w_0 = f(0) \in Q_0 \in \mathcal{W}$. The results of Section 4 show that $S \approx S(w_0)$, so it suffices to show that $C(f) \lesssim S(w_0)$.

Fix $z \in \mathbb{D} \setminus \{0\}$. Let $Q_{f(z)}$ be a Whitney square containing $f(z)$. Consider the point $\widetilde{z}$ with $|\widetilde{z}| = |z|$ and whose argument is the same as that of either one of the two endpoints of $\overline{C_z} \cap \partial\mathbb{D}$. Clearly $\rho_{\mathbb{D}}(z, \widetilde{z}) < r_0$, for some absolute positive number r_0.

Next, we consider the following collection of Whitney squares:

$$\mathcal{W}_{f(z)} = \left\{Q \in \mathcal{W} : \rho_\Omega\left(Q, Q_{f(z)}\right) \le r_0\right\}.$$

Lemma 5.1 yields an absolute upper bound $n(r_0)$ for the cardinality of $\mathcal{W}_{f(z)}$. Observe that $f^{-1}(\cup\mathcal{W}_{f(z)})$ contains the (shorter) subarc between z and $\widetilde{z}$ of the circle centered at 0 and with radius $|z|$, along with its symmetric counterpart. It follows that the radii through $f^{-1}(\cup\mathcal{W}_{f(z)})$ must cover C_z. Hence,

$$f(C_z) \subset \bigcup_{Q \in \cup\mathcal{W}_{f(z)}} S_{w_0}(Q).$$

Finally, since each $Q \in \mathcal{W}_z$ satisfies

$$\operatorname{diam}(Q) \approx \operatorname{diam}(Q_{f(z)}) \approx \delta_\Omega(f(z)),$$

we can conclude that

$$\frac{|f(C_z)|}{\delta_\Omega(f(z))} \lesssim \sum_{Q\in\mathcal{W}_z} \frac{|S_{w_0}(Q)|}{\operatorname{diam}(Q)} \le n(r_0)S(w_0) \lesssim S(w_0),$$

as desired.

It remains to show that if $g : \mathbb{D} \to \Omega$ is any Riemann mapping with $C(g) < \infty$ and φ is a conformal automorphism of $\mathbb{D}$, then $C(g\circ\varphi) < \infty$ as well. By using the distortion inequalities (5) as above, the assumption of the finiteness of $C(g)$ can be rewritten as

$$\sup_{z\in\mathbb{D}\setminus\{0\}} \frac{|g(C_z)|}{\delta_\Omega(g(z))} < \infty. \tag{7}$$

We wish to show that the corresponding supremum obtained by replacing g by $g\circ\varphi$ is also finite. Since the supremum in (7) would not change if we precomposed g with a rotation, we may assume that $\varphi(0) \neq 0$. We need only consider a point $z \in \mathbb{D}$ satisfying $|z| > |\varphi^{-1}(0)|$. Let $L = \partial C_z \cap \mathbb{D}$ denote the hyperbolic line which bounds C_z, and let $\widetilde{z}$ denote the point on the hyperbolic line $\varphi(L)$ which is closest to 0. Observe that $\varphi(C_z) = C_{\widetilde{z}}$.

Claim. $\rho_{\mathbb{D}}(\widetilde{z}, \varphi(z)) \le \rho_{\mathbb{D}}(0, \varphi(0))$.

The Claim follows from Lemma 2.1, in which we take the two hyperbolic lines to be determined by the pairs of points $\{0, \widetilde{z}\}$ and $\{\varphi(0), \varphi(z)\}$. Note that the hyperbolic line L is the unique hyperbolic line which is orthogonal to this pair of hyperbolic lines.

In light of the Claim, the distortion theorems can be used as in the proof of Theorem A to get that

$$\delta_\Omega(g(\varphi(z)) \approx \delta_\Omega(g(\widetilde{z})).$$

We conclude that the quotient $|g(\varphi(C_z))|/\delta_\Omega(g(\varphi(z)))$ is dominated by an absolute contant times the supremum in (7).

Remark. *In the last part of the above proof observe that for $|z| < |\varphi^{-1}(0)|$, we have $\rho_{\mathbb{D}}(0, z) \le \rho_{\mathbb{D}}(0, \varphi(0))$. Hence again by the distortion theorems we would have $\delta_\Omega(g(\varphi(z))) \approx \delta_\Omega(g(0))$. By analyzing the comparability constants we are led to the following quantitative estimate:*

$$C(g\circ\varphi) \lesssim \frac{C(g)}{(1-|\varphi(0)|)^2}.$$

The exponent 2 comes from Lemma 2.4.

5. Concluding Remarks

There are several interesting related questions one could ask with regard to the p-Poincaré inequality (1) when $p > 1$. First of all, one might wonder if a simple geometric characterization for the p-Poincaré inequality could be given, analogous to Theorem A, for simply connected planar domains. This problem was addressed in [SS-87] for the case $p = 2$ and [SS-90] for general $p \geq 1$. Several other authors have also studied this problem, see in particular [EH] and [Mar].

In [SS-87], the finiteness of the quantity

$$G_2 = \sup \left\{ \frac{|\Omega_1(\gamma)|}{\rho_\Omega(z_0, \gamma)} : \gamma \text{ is a hyperbolic geodesic crosscut of } \Omega \right\},$$

which corresponds to the quantity G of Theorem A, is shown to be a *necessary* but *not* sufficient condition for the 2-Poincaré inequality. A similiar situation holds for all $p > 1$.

The necessity result referred to above actually follows from the second author's characterization of Carleson measures for the Dirichlet Space in [Steg]. We can rewrite the finiteness of G_2 as the following inequality

$$\int_{C_z} |g'(\zeta)|^2 \, dA(\zeta) \leq \widetilde{C}(g) \frac{1}{\log \dfrac{1}{1 - |z|}}, \tag{8}$$

which is analogous to (3), for a conformal mapping g mapping the disk onto Ω. Since a 1-Poincaré domain is a 2-Poincaré domain, it follows that (3) implies (8), a fact that does not seem apparent from function theoretic considerations.

The Poincaré inequality (1) has a corresponding formulation where the Sobolev test functions are restricted to be holomorphic on Ω. In [Ham], Hamilton shows that this *analytic* p-Poincaré inequality is equivalent to the usual p-Poincaré inequality provided $p > 1$. His proof does not extend to the case $p = 1$, however, the authors believe that the result in the case $p = 1$ is true and further that it might be possible to use Theorem B to prove this.

Once a domain Ω is known to be a p-Poincaré domain, it is often useful to have estimates on the best constant $K_p(\Omega)$ for the p-Poincaré inequality. For a 2-Poincaré domain, the constant $K_2(\Omega)$ corresponds to the square root of the reciprocal of the smallest eigenvalue for the Neumann problem mentioned in the Introduction, and in fact a corresponding eigenfunction will serve as an extremal test function for the 2-Poincaré inequality. This approach has led to the calculation of the constants $K_2(\Omega)$ for an assortment of domains through PDE techniques. For $p \neq 2$ there is no such spectral theoretic formulation and much less is known. In his doctoral dissertation [Stan-90] the first author has determined the $K_p(\Omega)$ for all p in the case that Ω is a (bounded) interval in $\mathbb{R}^1$. It would be interesting to develop a method to exactly determine the constants $K_p(\Omega)$ ($p \neq 2$) for some planar domains, e.g., rectangles. For more on these matters, see [Stan-92].

We now give a simple application of Theorem A. Another important class of domains in the theory of partial differential equations is the class of *John domains*, which was introduced by John (under different terminology) in his work [John] on rotation and strain. A domain $D \subset \mathbb{R}^n$ is called a John domain if there exists a positive constant a along with a point $x_0 \in D$ (a *John center* of D) such that for each $x \in D$ there is a rectifiable path $\tau : [0, \ell(\tau)] \to D$ (parameterized with respect to arc length) such that $\tau(0) = x$, $\tau(\ell(\tau)) = x_0$, and $d(\tau(s), \partial D) \geq as$, for each $s \in [0, \ell(\tau)]$. A John domain is always a p-Poincaré domain for each p, $1 \leq p < \infty$. This was proved by Martio [Mar]. Näkki and Väisälä ([NV]) obtained a geometric characterization for (what they called) "John disks", that is, simply connected planar domains which are John domains. They showed that Ω is a John disk if and only if the following supremum

$$\sup\left\{\frac{\min\left(\operatorname{diam}\,[\Omega_1(\sigma)],\operatorname{diam}\,[\Omega_2(\sigma)]\right)}{\ell(\sigma)} : \sigma \text{ is a segmental crosscut of } \Omega\right\},$$

is finite. Noting that John domains are bounded, it immediately follows from Theorem A that John disks are Poincaré disks.

The converse of this fact is false. As a final application we give a class of examples which will demonstrate this and also show that the boundary of a Poincaré disk can be quite complicated. Let Ω be obtained from the open unit disk by removing any closed set of radial slits not containing the origin. Then, Ω is a simply connected domain and clearly L is finite. Such a domain need not, for example, be a John domain nor is the Riemann mapping function necessarily uniformly continuous. More generally, one can show that L is finite for any bounded starlike planar domain thereby giving a new proof that such domains are Poincaré domains (see [AS] for $p = 2$ and Theorem 6 in [SS-90] for general p).

References

[Ada] Adams, R.A., *Sobolev Spaces*, Academic Press, New York, NY, (1975).

[Ahl] Ahlfors, L.V., *Conformal Invariants*, McGraw-Hill, New York, (1973).

[AS] Axler, S. and Shields A., *Univalent multipliers of the Dirichlet space*, Michigan Math. J. **32** (1985), 65–80.

[Bea] Beardon, A.F., *The Geometry of Discrete Groups*, Springer Verlag, New York, (1983).

[EH] W.D. Evans and D.J. Harris, *Sobolev embeddings for generalized rigid domains*, Proc. London Math. Soc (3) **54** (1987), 141–175.

[GH] Gehring, F. W. and Hayman, W. K., *An inequality in the theory of conformal mapping*, J. Math. Pures Appl. **41** (1962), 353–361.

[GP] Gehring, F. W. and Palka, B. P., *Quasiconformally homogeneous domains*, J. Analyse Math. **30** (1976), 172–199.

[GT] Gilbarg, D. and Trudinger, N.S., *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin Heidelberg, (1983).

[HKM] Heinonen, J., Kilpeläinen, T., and Martio, O., *Nonlinear Potential Theory of Degenerate Elliptic Equations*, Oxford University Press, in press.

[John] John, F., *Rotation and strain*, Comm. Pure Appl. Math. **14** (1961), 391–413.

[Mar] Martio, O., *John domains, bilipschitz balls and the Poincaré inequality*, Rev. Roumaine Math. Pures Appl. **33:1-2** (1988), 107–112.

[Maz-68] Maz'ja, V. G., *On Neumann's problem in domains with nonregular boundaries*, Siberian Math. J. **9** (1968), 990–1012.

[Maz-85] ———, *Sobolev Spaces*, Springer-Verlag, Berlin Heidelberg, (1985).

[MS] Meyers, N. and Serrin, J., $H = W$, Proc. Nat. Acad. Sci. **51**, 1055–1056.

[NV] Näkki, R. and Väisälä, J., *John Disks*, preprint.

[New] Newman, M. H. A., *Elements of the Theory of Plane Sets of Points, Second Edition*, Cambridge University Press, Cambridge, UK, 1951.

[Pomm] Pommerenke, Ch., *Univalent Functions*, Vanderhoeck & Ruprecht, Göttingen, 1975.

[SS-87] Smith, W. and Stegenga, D. A., *Poincaré domains in the plane*, in *Complex Analysis*, Joensuu-Proceedings of R. Nevalinna Colloquium, Springer-Verlag Lecture Notes in Mathematics #1351 (1987), 312–327.

[SS-90] ———, *Hölder domains and Poincaré domains*, Trans. Amer. Math. Soc. **319** (1990), 67–100.

[Stan-90] Stanoyevitch, A., *Geometry of Poincaré Domains*, Doctoral Dissertation, University of Michigan, 1990.

[Stan-92] ———, *Products of Poincaré domains*, Proc. Amer. Math. Soc., To appear.

[Steg] Stegenga, D. A., *Multpliers of the Dirichlet Space*, Illinois J. Math. **24** (1980), 113–139.

[Ste] Stein, E. M., *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton, NJ, 1970.

[Zie] Ziemer, W. P., *Weakly Differentiable Functions* (1989), Springer-Verlag GTM#120, New York.

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