ARCLENGTH FORMULAS IN CONFORMAL MAPPING

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INTRODUCTION: Let $\Omega \subseteq \mathbb{C}$ be a Jordan domain (i.e., a bounded planar domain whose boundary is a Jordan curve) and let $\mathbb{D}$ denote the unit disk. Throughout we let "ℓ" denote arclength or more specifically the Hausdorff one-dimensional measure. We consider a homeomorphism $\varphi : \overline{\mathbb{D}} \longrightarrow \overline{\Omega}$ which is C^1 on $\mathbb{D}$. Given an arc $J \subseteq \partial\mathbb{D}$, we seek to express the length of the image arc: $\ell(\varphi(J))$ in terms of integral formulas involving (first-order) derivatives of φ. In this general situation, it is always true and straightfoward to prove that arclength is lower semicontinuous, i.e.,

$$\ell(\varphi(J)) \leq \varliminf_{r\uparrow 1} \ell(\varphi(rJ)) = \varliminf_{r\uparrow 1} \int_J \left| \frac{\partial}{\partial\theta} \varphi(re^{i\theta}) \right| d\theta \tag{1}$$

(here rJ denotes the dilation of $J : \{re^{i\theta} : e^{i\theta} \in J\}$).

The first progress on this problem when φ is conformal on $\mathbb{D}$ dates back to 1916 when F. & M. Riesz **[Rz-16]** proved that in case $\partial\Omega$ is rectifiable we then have the following formula valid for each arc $J \subseteq \partial\mathbb{D}$:

$$\ell(\varphi(J)) = \int_J \lim_{r\uparrow 1} \left|\varphi'(re^{i\theta})\right| d\theta \quad , \tag{2}$$

The question arises whether the hypothesis of rectifiability of $\partial\Omega$ in the Riesz result (2) can be weakened. By a result of Gehring and Hayman **[Ge&Ha–62]** if the image arc $\varphi(J)$ is rectifiable then so will be the image of the hyperbolic geodesic in $\mathbb{D}$ which joins the endpoints of J. The resulting Jordan subdomain of $\varphi(\mathbb{D})$ has rectifiable boundary and with an appropriate change of variables the Riesz result (2) can be shown to remain valid if only $\ell(\varphi(J)) < \infty$. With a bit more work, making use of the nontangential maximal function, one can go on to show that

$$\ell(\varphi(J)) = \lim_{r\uparrow 1} \ell(\varphi(rJ)) = \lim_{r\uparrow 1} \int_J \left|\varphi'(re^{i\theta})\right| d\theta \quad . \tag{3}$$

By (1), (3) is certainly true when $\ell(\varphi(J)) = \infty$ hence (3) is always valid, i.e., when φ is conformal, arclength is continuous.

In attempting to extend (2) to the remaining case $\ell(\varphi(J)) = \infty$, one problem is that the limits in (2) can fail to exist almost everywhere on J. In fact, as was

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shown by Lohwater, Piranian & Rudin **[LP&R–55]** the behavior of φ can be so extreme that for a.e. θ

$$\overline{\lim_{r\uparrow 1}}\left|\varphi'(re^{i\theta})\right| = \infty \quad , \quad \underline{\lim_{r\uparrow 1}}\left|\varphi'(re^{i\theta})\right| = 0 \ ,$$

(also $\overline{\lim\limits_{r\uparrow 1}}\ \arg\ \varphi'(re^{i\theta}) = \infty$, $\underline{\lim\limits_{r\uparrow 1}}\ \arg\ \varphi'(re^{i\theta}) = -\infty$). It is also true that any univalent function φ on $\mathbb{D}$ satisfies $\overline{\lim}_{r\uparrow 1}\left|\varphi'(re^{i\theta})\right| > 0$ a.e. Hence it seems plausible that if we replace the limit in (2) with a limit supremum, the resulting formula:

$$\ell(\varphi(J)) = \int_J \overline{\lim_{r\uparrow 1}}\left|\varphi'(re^{i\theta})\right| d\theta \tag{4}$$

might always be valid. The main purpose of this note is to give an example of a domain for which the formula (4) fails. The boundary of the domain is of fractal type and our methods rely on the recent pioneering work of Makarov **[Mak–85]** on boundary behavior of conformal mappings.

The author would like to express his thanks to Chris Bishop who had suggested to him such an approach to an example. Previously the author had a less pathological example constructed directly by its series expansion.

One can construct examples to show that with the exception of (1), none of our formulas have valid analogues in case $\varphi \in C^\infty(\mathbb{D})$ (with or without the rectifiability of $\partial\Omega$). We close this introduction with a question.

Question. *Which of our formulas have analogues in case φ is quasi–conformal on $\mathbb{D}$?*

THE CONSTRUCTION: In this section we construct a Jordan domain $\tilde{\Omega}$ and a conformal map $h : \mathbb{D} \longrightarrow \tilde{\Omega}$ such that $\ell(\partial\tilde{\Omega}) = \infty$ but

$$\int\limits_{\partial\mathbb{D}} \overline{\lim_{\substack{z\longrightarrow\zeta \\ z\in\mathbb{D}}}}\,|h'(z)|\,|d\zeta| < \infty$$

This integral differs from the one in (4) in that the limit supremum is unrestricted as opposed to radial and thus the above map is *a fortiori* a counterexample to (4). For the requisite definitions of Hausdorff measures and dimension we cite **[Fa–85]**. The domain $\tilde{\Omega}$ will be obtained from the familiar (Van Koch) snowflake domain Ω which is constucted as follows: Let Ω^0 be an equilateral triangle of side length 1. To obtain Ω^1, we build 3 new equilateral triangles exterior to Ω^0 each having side length $\frac{1}{3}$ and sharing the middle $\frac{1}{3}$ segments of the different sides of Ω^0. The domain Ω^1 is defined to be the union of Ω^0 with these 3 new equilateral triangles. Note that Ω^1 has $3\cdot 4$ sides each of length 3^{-1}. We continue this construction so that the n^{th} generation we have a polygon Ω^n with $3\cdot 4^n$ sides of length 3^{-n} each. The snowflake domain Ω is defined to be $\lim \Omega^n = \cup\Omega_n$

The snowflake Ω has recently been the subject of several investigations. In 1982, Kaufman and Wu proved **[Ka&Wu–85]** that the harmonic measure of Ω is supported on a set of Hausdorff dimension strictly smaller than that $(\log 4/\log 3)$ of the boundary curve. Subsequently, Carleson **[Ca–85]** showed that the harmonic measure of Ω lives on a set of Hausdorff dimension ≤ 1. Finally Makarov **[Ma–P]** showed that it lives on a set of length ($\equiv$ Hausdorff one–dimensional measure) zero. The techniques used to obtain these latter two results relied heavily on information and ergodic theory. We give an independent proof of the Makarov result based entirely on complex analysis which yields another property of the snowflake which we will use. Recall that a Plessner point of a meromorphic function $g : \mathbb{D} \longrightarrow \mathbb{C} \cup \{\infty\}$ is a point $\zeta \in \partial\mathbb{D}$ such that $\{g(z) : r < |z| < 1,\ z \in A_\zeta\}$ is dense in $\mathbb{C} \cup \{\infty\}$ for any $r < 1$ and Stolz angle A_ζ at ζ. Plessner proved **[Pl–27]** that the nontangential limit $\lim\limits_{\substack{z \longrightarrow \zeta \\ z \in A_\zeta}} g(z)$ exists in $\mathbb{C}$ at a.e. non-Plessner points $\zeta \in \partial\mathbb{D}$.

Lemma. *The harmonic measure of the snowflake Ω is supported on a set of length zero on $\partial\Omega$. Moreover, if $f : \mathbb{D} \longrightarrow \Omega$ is a Riemann map then a.e. $\zeta \in \partial\mathbb{D}$ is a Plessner point of f'.*

Proof. A theorem of Makarov states that for any Riemann map $g : \mathbb{D} \longrightarrow D$ which is so pathological that the nontangential (or radial) limits of g' exist in $\mathbb{C}$ almost nowhere, the harmonic measure of D will be supported on a set of zero length (i.e. there exists $A_0 \subseteq \partial\mathbb{D}$ such that $\ell(A_0) = 2\pi$ but $\ell(g(A_0)) = 0$; see **[Ma–84]** Theorem 3 or **[Po–86]** Corollary 1). We define $A_T = \{\zeta \in \partial\mathbb{D} : \lim_{r\uparrow 1} f'(r\zeta)$ exists in $\mathbb{C}\}$. The function f will necessarily be conformal at each point $\zeta \in A_T$ (see Theorem 10.5 and the accompanying discussion in **[Po–75]**). Define a cone with vertex p and angle α to be a set of the form

$$\{p + re^{i\theta} : 0 < r < r_0, \quad \theta_0 < \theta < \theta_0 + \alpha\}.$$

The conformality condition means, in particular, that for each $\epsilon > 0$ and each image point $f(\zeta)$ ($\zeta \in A_T$) on the boundary $\partial\Omega$ of the snowflake, there will be a cone with vertex $f(\zeta)$ and angle $\pi - \epsilon$ which is entirely contained in Ω. For a subset $A \subset A_T$ it is known that

$$\ell(A) = 0 \Longleftrightarrow \ell(f(A)) = 0$$

(see **[Po–75]** Theorem 10.16). These facts together with the aforementioned theorem of Plessner show that the lemma will be proved as soon as we show that the set T of those points in $\partial\Omega$ which possess the above cone property is of zero length. Letting,

$$T_n = \{p \in \partial\Omega |\ \text{there exists a cone in } \Omega \text{ with vertex } p, \\ \text{angle } \frac{99}{100}\pi \text{ and diameter } \geq 10 \cdot 3^{-n}\}$$

it is clear that $T \subseteq \cup T_n$ so we need only show that $\ell(T_n) = 0$. We consider first the n^{th} generation Ω^n of the snowflake. The boundary $\partial\Omega^n$ is made up of congruent arcs which consist of 4 segments each of length 3^{-n} such that the outer two segments are colinear and the inner two segments form two sides of an equilateral triangle. For a given such configuration, points on the outer two segments might be on T_n but points on the inner two segments (or points of later generations arising from these segments) cannot (because there is no room for the cone). Similarly when we pass from Ω^n to Ω^{n+1}, each of the outer two segments gets replaced by 4 segments and the 2 middle segments of each must not contain points of T_n. We conclude that for any $m \geq n$,

$$\begin{aligned} \ell(T_n) &\leq \ell(T_n \cap \partial\Omega^m) \\ &\leq \ell(\partial\Omega^n)\left(\frac{2}{3}\right)^{m-n} \end{aligned}$$

Hence $\ell(T_n) = 0$, as desired.
Aside: This argument shows that the Hausdorff dimension of T is $\leq \log 2/\log 3$.

We are now prepared to define the domain $\tilde{\Omega}$. For each Plessner point $\zeta \in \partial\mathbb{D}$ for f there exists a sequence $z_n = z_n(\zeta)$ such that $z_n \longrightarrow \zeta$ in some fixed Stolz angle and $f'(z_n) \longrightarrow 0$. For each $z = z_n$ we construct a hyperbolic tent T_z with hyperbolic vertex z and base I_z. This means that T_z is the (smaller) subdomain of $\mathbb{D}$ formed by 2 circular arcs which meet $\partial\mathbb{D}$ at right angles and have z as a common end point, the base I_z is defined as $\partial\overline{T_z} \cap \partial\mathbb{D}$. By applying (the conformally invariant analogue of) Theorem 10.8 in **[Po–75]**, T_z can be made to have the following properties: ("$\approx$" means comparable and "$\lesssim$" shall mean "$\leq$" with a constant)

(i) $\zeta \in I_z$,
(ii) $\ell(\partial T_z) \approx \ell(I_z)$
(iii) $\displaystyle\int_{\partial T_z \cap \mathbb{D}} |f'(w)|\ |dw| \lesssim |f'(z)|(1 - |z|) \lesssim \ell(I_z)$.

Letting $F \subseteq \partial\mathbb{D}$ be a compact null set whose image $f(F)$ has Hausdorff dimension greater than 1 (the existence of F is guaranteed by the lemma since $\partial\Omega$ has Hausdorff dimension > 1), we note that for a Plessner point $\zeta \ \varepsilon\ \partial\mathbb{D} \sim F$, and for $z_n(\zeta)$ close enough to ζ we have $I_{z_n} \cap F = \varnothing$. Hence the collection $\langle I_{z_n}(\zeta)\rangle$ covers $\partial\mathbb{D} \sim F$ a.e. in the sense of Vitali so we can find a disjoint collection of such intervals $\langle Ij \rangle$ such that

(iv) $Ij \cap F = \varnothing$ for all j and
(v) $\ell(\cup Ij) = 2\pi$

Let $\tilde{D} = \mathbb{D} \sim \overline{\cup Tj}$ and $\tilde{\Omega} = f(\tilde{D})$ then $\dim(\partial\mathbb{D}) > 1$ since it contains $f(F)$. On the other hand, f' is continuous at each point of $\partial\tilde{D} \sim \partial\mathbb{D}$ and

$$\begin{aligned} \int_{\partial\tilde{D}} |f'(w)|\ |dw| &= \sum \int_{\partial T_j} |f'(w)|\ |dw| \\ &\lesssim \sum |f'(z_j)|(1 - |z_j|) \\ &\lesssim \sum \ell(I_j) \\ &= 2\pi < \infty\ . \end{aligned}$$

The desired Riemann map is now readily obtained by composing f with any Riemann map $k : \mathbb{D} \longrightarrow \tilde{D}$.

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