# Chapter 1: Logical Operators

## HELICOPTER TOUR

Logic is the foundational edifice on which mathematics, computer science, and philosophy all rest. The importance of being able to make valid and convincing arguments in many subjects gave early impetus to the development of logic in the ancient societies of Greece, China, and India. The Greek philosopher Aristotle (384–322 BC) is generally considered to be the founder of the subject. Over the years, protocols and computing languages have evolved and they will continue to change, but the logic governing how they are organized remains the same. The formal subject of **logic** aims to establish a coherent framework in which all scientific thoughts can be expressed, communicated, and synthesized. In this chapter we will cover the essential principles of logic that will be needed in order to understand and write theorems, proofs, algorithms, and programs. Most topics in mathematics and computer science will put the reader in constant need of logical principles and will continue to reinforce the reader's mastery of logical thinking and inferences.

## 1 Statements and Truth Values

Our first definition introduces the building blocks of logic; although it is not technically formulated it will suffice for our purposes.

**DEFINITION 1:** A **statement** is any declarative sentence or mathematical relation that has a **truth value** of either **true** or **false**.

**EXAMPLE 1:** For each item below, indicate whether it is a statement, and, if possible, indicate the truth value.

(a) Honolulu is the capital of Hawaii.
(b) What is your name?
(c) $-5 < 2$
(d) It is not possible to have $x^3 + y^3 = z^3$, for three nonzero *integers* $x$, $y$, and $z$.

**Note:** An **integer** is a real number with no decimal part, i.e., a number among the list $0, \pm 1, \pm 2, \cdots$.

SOLUTION: (a) is a true statement. (b) is not a statement. (c) is a true statement, as is (d). Do not worry if you did not know the truth value of (d); it is quite nontrivial.[1]

Item (d) motivates introducing some general terminology regarding statements in the sciences.

**DEFINITION 2:** A **theorem** or a **proposition** is a true statement that has been proved. Theorems are usually of greater significance than propositions. A **lemma** is a true statement that has been proved, but is usually intended to be used to prove other results (theorems or propositions), rather than being of interest in its own right. A **corollary** is a true statement that is rather easily seen as a consequence or an interesting special case of a deeper theorem or proposition. A **conjecture** is a statement that is believed to be true but has not yet been proved.

Note that a conjecture is a statement that has a truth value, even though this truth value is not known (at present). In the pure scientific fields the unspoken etiquette requires that all (nonconjecture) proclamations of statements need to be backed up with a proof. This is unfortunately not always the case in some other disciplines. Mathematics is a subject that is rich in open questions and conjectures, some dating back hundreds of years. Although there is no Nobel Prize in mathematics (the generally believed yet unsubstantiated reason for this is quite humorous), there are several valuable prizes that are awarded for mathematical achievement. Perhaps the most prestigious mathematical prizes are the *Fields medals*. Unlike the Nobel, Fields medals must be awarded to mathematicians under 40 years of age (so as to better motivate their future work). In 2000, The Clay Institute in Canada put out a set of seven very difficult *Millenium Problems*. Each problem carries a US$1 million award to the first person to solve it. At the time of this writing so far, one of these seven problems has been solved. This problem, known as *the Poincaré conjecture*, was solved in 2003 by the Russian mathematician Grigori Perelman. Despite living an extremely modest lifestyle (sharing a small one-bedroom Moscow apartment with his mother), Perelman declined the prize money, humbly justifying his decision by indicating that several other previous mathematicians deserve most of the credit for having primed the apparatus for his proof.

## 2 Negations, Conjunctions, and Disjunctions

New statements can be obtained by combining other statements using **logical operators** (or **logical connectors**); moreover, truth values of such new statements can be inferred from the truth values of the statements from which they are constructed. Logical inference basically refers to the protocol of writing correct proofs, and thus lies at the foundation of all scientific theories. We begin by introducing the most basic logical operators.

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[1] This statement is part of *Fermat's Last Theorem*, named after the French barrister and amateur mathematician Pierre de Fermat, who scribbled his statement in the margin of a book in the 1630s, where he claimed that there was not enough space to contain his "remarkable proof." The complete statement reads that if $n$ is any positive integer greater than two, then the equation $x^n + y^n = z^n$ can have no nonzero integer solutions: $x, y, z$. This "theorem" remained unproved until the mid 1990s (to be a true statement) by Princeton mathematician Andrew Wiles, who realized a childhood dream. He had spent seven years secretly working feverishly (in his attic) on this problem. See [Kle-00] for a very interesting historical account of this famous theorem.

**DEFINITION 3:** Suppose that $P$ and $Q$ represent statements.
(i) The **negation** of $P$, denoted $\sim P$, and read as "not $P$," is another statement whose truth value is the opposite of that of $P$. Thus, $\sim P$ is false when $P$ is true and $\sim P$ is true when $P$ is false.
(ii) The **conjunction** of $P$ and $Q$, denoted $P \wedge Q$, and read as "$P$ and $Q$," is another statement that will be true exactly when both $P$ and $Q$ are true (and so will be false in all other cases).
(iii) The **disjunction** of $P$ and $Q$, denoted $P \vee Q$, and read as "$P$ or $Q$," is another statement that will be true as long as $P$ is true, or $Q$ is true (or both).

Some comments are in order. These definitions agree with those of formal (and contractual) English. In spoken English, the words "not" and "and" are always clear, but there are several ways to form a sentence that use them. For example, the negation of the statement: "You are rich" is usually not expressed in the formal way: "It is not the case that you are rich," but rather as "you are not rich." Caveat: "You are poor" is not the negation of "You are rich." (Why?) We point out that in this context "but" is a synonym for "and."

The word "or," however, is sometimes ambiguous because in certain cases it is not intended that both $P$ and $Q$ can be true. For example, suppose a restaurant waiter asks you: "would you like the soup or the salad that comes with the meal?" If you ask for both, you will either get a snappy retort from the waiter to choose only one, or an unexpected extra charge on your bill. The sciences (and formal English) cannot tolerate such ambiguity, so by default the word "or" has the meaning as in (iii) above. In case one wants to use the more restrictive disjunction of two statements $P$ and $Q$, one calls it the **exclusive disjunction**, and we denote it as $P \oplus Q$. This is read simply as "$P$ or $Q$ but not both." Any statement that is made using logical connectives and other more basic statements is called a **compound statement**.

## 3 Truth Tables

A **truth table** for a compound statement is a complete listing of all of the possible cases of the truth values for the basic statements from which the compound statement is constructed (thought of as *logical variables* of the compound statement) along with the resulting truth values for the compound statement. Tables 1, 2, and 3 below give truth tables for the negation, conjunction, and disjunction logical connectors.

| $P$ | $\sim P$ |
|---|---|
| T | F |
| F | T |

| $P$ | $Q$ | $P \wedge Q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | F |

| $P$ | $Q$ | $P \vee Q$ |
|---|---|---|
| T | T | T |
| T | F | T |
| F | T | T |
| F | F | F |

**TABLES 1–3:** Truth tables for a negation, conjunction, and disjunction.

If we had to make a truth table involving three (variable) basic statements, say $(P \wedge Q) \vee R$, the truth table would require eight rows of truth values. The first four rows would have $P$ = T(rue), the last four rows would have $P$ = F(alse); the $Q$ logical variable's column would alternate in pairs: T,T,F,F,T,T,F,F, and the final $R$ variable's column would alternate in single truth values: T,F,T,F,T,F,T,F. With this organization, all possible combinations of truth values of the variables are accounted for. In general, a truth table for a compound statement involving $n$ basic statements will contain $2^n$ rows of truth values. This is an easy consequence of the *multiplication principle* that will be introduced in Chapter 11, but it would behoove readers who have not yet seen this principle to directly convince themselves of this fact.

## 4 Implications

We next come to the very important logical connective of *implication* that will allow logical inferences. Implications very often get abused (and misunderstood) in spoken English, and furthermore, there are many different ways of expressing them both in written and in spoken language. Consequently it may take some time to get comfortable with them. Before we give the formal definition, we will warm up with the following nontechnical example of an implication:

Suppose that Professor Saunders tells his student Jimmy: "Jimmy, if you get at least a 90 on the final, then I will give you an A for the course." This is a compound statement of the form: If $P$, then $Q$, where $P$ = "Jimmy gets at least a 90 on the final," and $Q$ = "Professor Saunders gives Jimmy an A for the course." As with all compound statements, the truth value of the (whole) implication will be determined by the individual truth values of $P$ and $Q$.

Most people would agree that the implication (if $P$ then $Q$) is true when both $P$ and $Q$ are true (Jimmy gets at least a 90 on the final and Prof. S. gives him an A), and false when $P$ is true but $Q$ is false (because if Jimmy gets at least a 90 on the final, but Prof. S. does not give him an A, then Prof. S. has broken his promise). These truth values are indeed logically correct.

The remaining cases are often ambiguous and not well understood in spoken language. For example, what would be the truth value of this implication in case $P$ is false, and $Q$ is true? This would mean that Jimmy did not get at least a 90 on the final, but Prof. S. still gave Jimmy an A. Would this make our implication (If $P$, then $Q$) true or false? In this case, the implication is true. One helpful way to understand this is to view the implication as a promise or guarantee. As long as the promise is not broken, the implication is true. Thus, if Jimmy did not get at least a 90 on his exam, Prof. S. has no obligation to give Jimmy an A. Whether he does or does not, his promise would not be broken. He may have decided to give Jimmy an A for an assortment of reasons (perhaps the exam was more difficult than Prof. S. had anticipated, and the average score was only 52). In the same fashion, the implication "If $P$, then $Q$" is true if both $P$ and $Q$ are false—here, Jimmy fails to get at least a 90 on the exam, and Prof. S. does not give Jimmy an A. Again, Professor Saunders' promise to Jimmy is not broken.

**DEFINITION 4:** Suppose that $P$ and $Q$ represent statements. The **implication** or **conditional statement** $P \rightarrow Q$, which can be read as "If $P$, then $Q$," is another statement that will be true in all cases, except when $P$ is true and $Q$ is false. In an implication $P \rightarrow Q$, $P$ is called the **hypothesis**, and $Q$ is called the **conclusion**.

The truth table for the implication is shown in Table 4.

| $P$ | $Q$ | $P \to Q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

**TABLE 4:** Truth table for an implication.

Unlike with the other logical connectors that were previously introduced, the truth values of a conditional statement are much less obviously related to spoken English. For example, note that the conditional $P \to Q$ will <u>always</u> be true if the hypothesis $P$ is false. Thus, a nonsensical English statement such as, "If the moon is made of blue cheese, then Germany won World War II," would be a true statement (as far as logic is concerned), since both the hypothesis and conclusion are false (see row four of Table 4).

There are numerous ways of expressing conditional statements both in written and spoken language, especially in the sciences where logical inference is central to the subjects. Here are several common variants of wording the implication "if *P*, then *Q*" $(P \to Q)$:

1. "*P* is sufficient for *Q*" (or "*P* is a sufficient condition for *Q*").
2. "*Q* is necessary for *P*" (or "*Q* is a necessary condition for *P*").
3. "*P* implies *Q*" (or "*Q* is implied by *P*") (or "*Q* follows from *P*").
4. "*P* only if *Q*."[2]
5. "*Q*, if *P*."

## 5 Converses and Contrapositives

In order to be able to understand books, papers, and lectures in any scientific discipline, you must become familiar with the different ways of expressing a conditional statement. In particular, you must take care to distinguish between the hypothesis and the conclusion of a conditional; if they are interchanged, you get a different conditional, as we will now see.

**DEFINITION 5:** Given an implication $P \to Q$, the **converse** is the implication $Q \to P$, obtained by interchanging the hypothesis and conclusion. If, in addition, we also negate both the hypothesis and conclusion, we arrive at the so-called **contrapositive** of the implication $P \to Q$: $\sim Q \to \sim P$.[3]

For example, the implication: "if Spot is a Doberman pinscher, then Spot is a dog," is (always) a true statement since a Doberman pinscher is a special breed of dog. The converse, "if Spot is a dog, then Spot is a Doberman pinscher," however, could be false, for example, if Spot were a collie

[2] To help remember this one, you should view "only if" as a synonym for "then."
[3] To make this more clear, we could write $(\sim Q) \to (\sim P)$, but in logic negations get done first, so the parentheses are not necessary.

(that would put us in line two of Table 4). On the other hand, if we form the contrapositive: "if Spot is not a dog, then Spot is not a Doberman pinscher," we get something that makes good sense and is in fact a true statement. This is no coincidence and Part (a) of the following example show why this is true in general.

**EXAMPLE 2:** (a) Construct a truth table for the and its contrapositive $\sim Q \to \sim P$, and then compare with that for the original implication $P \to Q$.
(b) Construct a compound statement with one (logical) variable that is always true.
(c) Construct a compound statement with two (logical) variables that is always false.

| $P$ | $Q$ | $\sim Q \to \sim P$ | | |
|---|---|---|---|---|
| T | T | F | T | F |
| T | F | T | F | F |
| F | T | F | T | T |
| F | F | T | T | T |

SOLUTION: (a) The following truth table for $\sim Q \to \sim P$ will illustrate a general method for creating truth tables of compound statements using what we know about the basic logical connectors: The first two columns (for the two variables) are constructed in the usual way with four rows total. For the three columns under the compound statement $\sim Q \to \sim P$, the outer two (unshaded) columns are done first for the negations $\sim Q$ and $\sim P$. These are simply the opposite truth values as those in the $Q$ and $P$ columns. The shaded column's truth values (those for the contrapositive $\sim Q \to \sim P$) are then obtained by remembering that an implication is true in all cases except when the hypothesis ($\sim Q$ on left) is T and the conclusion ($\sim P$ on right) is F.

If we delete the two unshaded columns below $\sim Q \to \sim P$ that we used to help us get the shaded truth values, we get the same truth table as that for the implication $P \to Q$ (shown in Table 4).

(b) Recall that a disjunction $P \vee Q$ will be true in all (three out of four) cases except with both parts $P$ and $Q$ are F(alse). Since $P$ and $\sim P$ will always have opposite true values, one of them will always be true. Thus, $P \vee \sim P$ (or $P \vee (\sim P)$ ) will always have truth value T(rue). A truth table (with two rows) could also be used to verify this.

(c) Recall that a conjunction $P \wedge Q$ will be false in all (three out of four) cases except with both parts $P$ and $Q$ are T(rue). It follows (as in part (b)) that $P \wedge \sim P$ will always be false. But we were asked to find a compound statement with two variables. We can simply use $(P \wedge \sim P) \wedge Q$. (Since the first part is always false, the conjunction with $Q$ will be false no matter if $Q$ is true or false.)

## 6 Logical Equivalence and Biconditionals

The previous example nicely motivates the three concepts in the following definition. In part (a) of Example 2, we discovered that the truth tables for and implication $P \to Q$ and its contrapositive $\sim Q \to \sim P$ are the same. Parts (b) and (c) gave examples of compound statements that are true, no matter which values the logical variables take on.

**DEFINITION 6:** Two compound statements A, B are said to be (**logically**) **equivalent** (written as A $\equiv$ B or as A $\Leftrightarrow$ B) if their truth tables are identical. [4] A compound statement that is always true (regardless of the values of its logical variables) is called a **tautology**; if it is always false, it is called a **contradiction**.

Thus we have discovered that an implication is logically equivalent to its contrapositive: in symbols:

$$\sim Q \to \sim P \quad \equiv \quad P \to Q$$

Constructing truth tables for two compound statements can be used to show they are logically equivalent. Sometimes more insightful proofs of a logical equivalence can be done using logical reasoning rather than by truth tables. Here is such of proof of the above logical equivalence:

The only situation where the contrapositive $\sim Q \to \sim P$ will be false is when the hypothesis ($\sim Q$) is true and the conclusion ($\sim P$) is false. This is equivalent to $P$ being true and $Q$ being false, which is precisely the only situation when the implication $P \to Q$ is false. Thus the two are logically equivalent. Such proofs require more thinking than grinding through a truth table, but they often shed more light on the concept and they can sometimes be faster.

Logical equivalences are to logic what identities are in trigonometry: both sides are the same no matter the value(s) of the variables involved. In Example 2 we also discovered some examples of tautologies: $P \vee \sim P$ and of contradictions: $P \wedge \sim P$

**EXAMPLE 3:** (a) Find a compound statement involving only logical operators from the list $\wedge, \vee, \sim$, that is logically equivalent to the implication $P \to Q$.
(b) Find a compound statement involving only the conjunction $(\wedge)$ and negation $(\sim)$ operators that is logically equivalent to the disjunction $P \vee Q$.

SOLUTION: (a) Implication (Table 4) and disjunction (Table 3) have one important property in common: their truth values are true in exactly three out of the four possible cases. Recall that a disjunction is true, except when both variables are false. On the other hand, the implication $P \to Q$ is true, except when $P$ is true and $Q$ is false, or equivalently, except when $\sim P$ and $Q$ are both false. Putting this all together, we conclude that $P \to Q$ and $\sim P \vee Q$ are logically equivalent. A truth table can always be used to check any purported equivalence, and any skeptical readers are encouraged to do this. Notice that we did not use conjunction.
(b) The basic strategy to discover such an equivalence is this: A disjunction $P \vee Q$

$\sim(\sim P \wedge \sim Q)$. This can be checked with truth tables. Alternatively: The conjunction $\sim P \wedge \sim Q$ is false except in the case both parts $\sim P$ and $\sim Q$ are false, which is the same as both $P$

[4] The concept of logical equivalence can be extended to cases where one of A or B has more logical variables than the other, if these additional variables are redundant. For example, if A and B were tautologies we would certainly have A $\equiv$ B.

and $Q$ are true. So it's negation will be true in all cases except when both $P$ and $Q$ are false—exactly as in the definition of the disjunction $P \vee Q$."

Abuses of conditionals in spoken language are rampant. For example, suppose that a parent tells his or her son: "If you don't finish your homework, then you will not be allowed to go out tonight." We know this statement is equivalent to its contrapositive: "If you are allowed to go out tonight, then you will have finished your homework." This latter statement would be true if the son were not allowed to go out tonight even though he finishes his homework $(F \to T)$, but this is certainly not what the parent intended. The parent had really intended that the converse also be true: "If you finish your homework, then you will be allowed to go out tonight."[5] In other words, the parent intended (and the son surely understood) that the truth values of the two parts of this statement should be the same. Thus the intended and interpreted statement is not a conditional, but rather a so-called *biconditional statement*; we enunciate this important logical construction in the following definition.

**DEFINITION 7:** Suppose that $P$ and $Q$ represent statements. The **biconditional statement** $P \leftrightarrow Q$, which can be read as "$P$ if, and only if $Q$," is another statement that is true whenever $P$ and $Q$ share the same truth values, and false when $P$ and $Q$ have opposite truth values.

The truth table for the biconditional is shown in Table 5. As pointed out before the definition, biconditionals are often intended and interpreted when a conditional statement is given in spoken language; indeed, you rarely hear the "if and only if" phrase in spoken language. An often-used variation (in formal written language) of "$P$ if, and only if $Q$" is "$P$ is (both) necessary and sufficient for $Q$."

| $P$ | $Q$ | $P \leftrightarrow Q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | T |

**TABLE 5:** Truth table for a biconditional.

It follows that the logical equivalence of two compound statements $\mathsf{A}$, $\mathsf{B}$ ($\mathsf{A} \equiv \mathsf{B}$) can be expressed as saying $\mathsf{A} \leftrightarrow \mathsf{B}$ is a tautology. (Reason: That the biconditional is a tautology means that both sides $\mathsf{A}$, $\mathsf{B}$ will always have the same truth values, whatever the truth values of the logical variables involved $P$,$Q$, etc.)

## 7 Hierarchy of Logical Operators

We have already pointed out that in any compound statement, negations always get done first. As with arithmetic operations, there is a convention of a **hierarchy** of logical operations, but parentheses can always overrule any such hierarchy. The precedence is as shown in Table 6:

[5] This is the converse of the contrapositive. Since the contrapositive is equivalent to the original statement, its converse is also equivalent to that of the original statement.

| OPERATOR | HIERARCHY |
|---|---|
| $\sim$ | highest (do first) |
| $\wedge$ | next highest |
| $\vee$ | third highest |
| $\rightarrow$ | lower |
| $\leftrightarrow$ | lowest (do last) |

**TABLE 6:** Hierarchy of logical operators; parentheses can overrule any hierarchy.

The hierarchy conventions of Table 6 are standard, and help to avoid unnecessary parentheses. For example, compare the following compound statement with and without redundant parentheses:[6]

$$(P \vee (Q \rightarrow (\sim R))) \rightarrow [(\sim Q) \wedge P] \equiv P \vee (Q \rightarrow \sim R) \rightarrow \ \sim Q \wedge P.$$

Only one of the five sets of parentheses is actually needed.[7]

Truth tables for compound statements can be obtained by adding separate new columns for logical statements that build up to the final statement. The method illustrated Example 2(a) is usually faster though. For completeness, we give an example of building a truth table using this new method.

**EXAMPLE 4:** Construct a truth table for the statement: $(P \wedge Q) \rightarrow (Q \rightarrow R)$.

SOLUTION: Two versions of a truth table are shown in Table 7; the first one using the new method just explained, while the second is done in the more compact fashion that we introduced earlier. The basic idea in both is the same: we start by evaluating the logical operators highest in the hierarchy first, eventually working our way to the last operators lowest in the hierarchy, whose truth values determine those of the given compound statement.

| $P$ | $Q$ | $R$ | $P \wedge Q$ | $Q \rightarrow R$ | $(P \wedge Q) \rightarrow (Q \rightarrow R)$ |
|---|---|---|---|---|---|
| T | T | T | T | T | T |
| T | T | F | T | F | F |
| T | F | T | F | T | T |
| T | F | F | F | T | T |
| F | T | T | F | T | T |
| F | T | F | F | F | T |
| F | F | T | F | T | T |
| F | F | F | F | T | T |

[6] As in arithmetic, logical operations of the same hierarchy are performed from left to right.
[7] Unlike in arithmetic, the hierarchy of Table 6 is not always so well-known by nonspecialists (of logic); we therefore will not always aim for minimizing parentheses usage in forming compound statements.

| $P$ | $Q$ | $R$ | $(P \wedge Q) \to (Q \to R)$ | | |
|---|---|---|---|---|---|
| T | T | T | T | T | T |
| T | T | F | T | F | F |
| T | F | T | F | T | T |
| T | F | F | F | T | T |
| F | T | T | F | T | T |
| F | T | F | F | T | F |
| F | F | T | F | T | T |
| F | F | F | F | T | T |

**TABLE 7:** Two versions of truth tables for $(P \wedge Q) \to (Q \to R)$.

## 8 Some Useful Logical Equivalences

A logical problem that often arises is to decide whether two compound statements A, B, involving the same logical variables, are logically equivalent. One may, of course, construct truth tables for both to see if they are identical, but it is helpful to know a few common logical equivalences. Some of the more useful logical equivalences are summarized in the following result:

**THEOREM 1:** (*Some Logical Equivalences*) We let $P$, $Q$, and $R$ denote any statements, **T** denote any tautology, and **F** denote any contradiction. The following logical equivalences are (always) valid:

PART I: Equivalences involving conjunctions, disjunctions, and, negations:
(a) (*Commutativity*) $P \wedge Q \equiv Q \wedge P$, $P \vee Q \equiv Q \vee P$

(b) (*Associativity*) $(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$, $(P \vee Q) \vee R \equiv P \vee (Q \vee R)$

(c) (*Distributivity*)
$$P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R), \ P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$$

(d) (*De Morgan's Laws*) $\sim (P \vee Q) \equiv \sim P \wedge \sim Q$, $\sim (P \wedge Q) \equiv \sim P \vee \sim Q$ [8]

(e) (*Double Negation*) $\sim (\sim P) \equiv P$

(f) (*Absorption*) $P \vee (P \wedge Q) \equiv P$, $P \wedge (P \vee Q) \equiv P$

(g) (*Identity Laws*) $P \wedge \mathrm{T} \equiv P$, $P \vee \mathrm{F} \equiv P$

PART II: Equivalences involving conditionals and biconditionals:
(a) (*Implication as Disjunction*) $P \to Q \equiv \sim P \vee Q$

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[8] The following observations can help to make the very important De Morgan's laws a bit easier to remember: They can be thought of as distributive laws for negation over disjunctions and conjunctions. The negation gets distributed, but the conjunction gets changed to a disjunction, and vice versa. For the first one: Since the disjunction $P \vee Q$ is true in three out of the four cases (see Table 1.3); its negation $\sim (P \vee Q)$ is false in three out of four cases, like a conjunction (see Table 1.2). In words, the formula $\sim (P \vee Q) \equiv \sim P \wedge \sim Q$ can be read as: "the negation of a disjunction is the conjunction of the negations." If we interchange the words disjunction/conjunction in this phrase, we get the wording for the second De Morgan law.

(b) (*Negation of Implication*) $\sim(P \to Q) \equiv P \wedge \sim Q$
(c) (*Exportation*) $P \to (Q \to R) \equiv (P \wedge Q) \to R$
(d) (*Contrapositive Equivalence*) $P \to Q \equiv \sim Q \to \sim P$
(e) (*Biconditional as Implications*) $P \leftrightarrow Q \equiv (P \to Q) \wedge (Q \to P)$

PART III: Some tautologies and contradictions:
(a) (*Tautologies*) $P \vee \mathrm{T} \equiv \mathrm{T}$, $P \vee \sim P \equiv \mathrm{T}$, $P \to \mathrm{T} \equiv \mathrm{T}$, $\mathrm{F} \to Q \equiv \mathrm{T}$
(b) (*Contradictions*) $P \wedge \mathrm{F} \equiv \mathrm{F}$, $P \wedge \sim P \equiv \mathrm{F}$

*Sketch of Proof:* Each of these equivalences can be verified using truth tables, and we have already verified some. Once a part has been proved it can be used in the proof of other parts. (More generally, mathematical theorems can be used to prove other theorems.) We will do some of the key parts here and leave some other to the Exercises.

Part I(e) (*Double Negation*) $\sim(\sim P) \equiv P$ can either be checked with a truth table, or simply noting that since $\sim P$ has the opposite truth values of *P*, it follows that its negation $\sim(\sim P)$ will have the same truth values as *P*.

Part I(d) (*De Morgan's Laws*) Recall that in Example 3(b), we discovered the logical equivalence: $\sim(\sim P \wedge \sim Q) \equiv P \vee Q$. It follows that the negations of both sides will also be logically equivalent $\sim(\sim(\sim P \wedge \sim Q)) \equiv \sim(P \vee Q)$. But using Part I(e) (that we just proved), we get that that $\sim(\sim(\sim P \wedge \sim Q)) \equiv \sim P \wedge \sim Q$. (In any logical identity, we can replace each logical variable with any logical compound statement; in I(e), we replaced P with $\sim P \wedge \sim Q$).) Putting the last two logical equivalences together gives us $\sim(P \vee Q) \equiv \sim P \wedge \sim Q$, which is the first De Morgan Law. The other De Morgan Law can be proved in a similar fashion or more easily (but less insightfully) verified using truth tables.

Part II(a): $P \to Q \equiv \sim P \vee Q$: We discovered this in Example 3(a).
Part II(b) $\sim(P \to Q) \equiv P \wedge \sim Q$:
First use II(a) and negate both sides: $\sim(P \to Q) \equiv \sim(\sim P \vee Q)$. Next we apply De Morgan's Law I(d) (the first one) to the negation on the right to write $\sim(\sim P \vee Q) \equiv \sim(\sim P) \wedge \sim Q$. Finally double negation I(e), shows the last expression is equivalent to $P \wedge \sim Q$, and putting this all together gives the desired equivalence II(b).

Such proofs are more elegant and valuable than a simple (but rote) pair of truth tables. The exercises will ask the reader to prove more of the above equivalences. The serious reader, however, would be advised to now go through proving each of the above statements, avoiding truth table proofs whenever possible.[9] □

We point out that Part II(b) $\sim(P \to Q) \equiv P \wedge \sim Q$ of Theorem 1, provides a practical template for negating an implication in spoken or written English. For example, to negate the implication: "If it rains, then I will go to see a movie," rather than the formal (and highly inelegant) "It is not the case that if it rains, then I will go to see a movie," we could express it as: "It will rain and I will not go

[9] It is fine (and all the better) to recognize which equivalences have previously been proved (and to skip them).

to see a movie." Note that this latter statement would always have the opposite truth values of the original implication.

## 9 Logical Implication

Recall that a logical equivalence $A \equiv B$ (or $A \Leftrightarrow B$) means that whenever the statement $A$ is true, so is $B$, and whenever $A$ is false, so is $B$, i.e., $A$ and $B$ have the same truth values in corresponding rows of their truth tables. If we take only the first half of this definition, i.e., whenever the statement $A$ is true, so is $B$, we arrive at the important concept of *logical implication*:

**DEFINITION 8:** If $A$, $B$ are two compound statements, we say that $A$ **(logically) implies** $B$ (written as $A \Rightarrow B$) if whenever the statement $A$ is true, so is $B$.

Thus to check an implication $A \Rightarrow B$, we need only look at situations (rows of the truth tables) where $A$ is true. If $B$ is ever false in such a situation (even just once), then the logical implication $A \Rightarrow B$ is invalid; otherwise it is valid.

We explained earlier that a logical equivalence $A \equiv B$ is valid if, and only if the compound statement $A \leftrightarrow B$ is a tautology. In a similar fashion, it can be easily seen that the logical implication $A \Rightarrow B$ is valid if, and only if the compound statement $A \to B$ is a tautology. The following result will summarize a few useful and important logical implications.

**THEOREM 2:** (*Some Logical Implications*) We let $P$, $Q$, and $R$ denote any statements, **T** denote any tautology, and **F** denote any contradiction. The following logical implications are (always) valid:

(a) (*Addition*) $P \Rightarrow P \vee Q$

(b) (*Subtraction*) $P \wedge Q \Rightarrow P$

(c) (*Modus Ponens*) $P \wedge (P \to Q) \Rightarrow Q$

(d) (*Modus Tollens*) $(P \to Q) \wedge \sim Q \Rightarrow \sim P$

(e) (*Hypothetical Syllogism*) $(P \to Q) \wedge (Q \to R) \Rightarrow P \to R$

(f) (*Disjunctive Syllogism*) $(P \vee Q) \wedge \sim P \Rightarrow Q$

(g) (*Constructive Dilemmas*) $(P \to Q) \wedge (R \to S) \Rightarrow [(P \vee R) \to (Q \vee S)]$
$(P \to Q) \wedge (R \to S) \Rightarrow [(P \wedge R) \to (Q \wedge S)]$

*Sketch of Proof:* At this point, each of these implications should seem quite reasonable to the reader. Each part can be proved by constructing truth tables for each side, and checking to see that in all cases where the first compound statement ($A$ ) is true, the second compound statement ($B$) is also true. Rows where $A$ is false can be ignored when checking the implication. We will try to promote more elegant modes of proof that will encourage a deeper understanding. Below we give proofs of only two parts of the theorem, leaving the rest as exercises. Rather than do easy and tedious truth tables, we will give more insightful logical arguments. Of course, when in doubt, truth tables can always be used.

Part (c): The only situation that could make $P \wedge (P \to Q) \Rightarrow Q$ false is if $Q$ were to be false and $P \wedge (P \to Q)$ were to be true. We will show that this cannot happen. The truth of the conjunction implies (by its definition, or by subtraction—Part (b) of Theorem 2 and also using commutativity: $P \wedge Q \equiv Q \wedge P$) that both the implication $P \to Q$ and its hypothesis $P$ are true. This forces the conclusion $Q$ of the implication to also be true because otherwise the implication would be false (i.e., it would have the form $T \to F$). This proves that the original implication $P \wedge (P \to Q) \Rightarrow Q$ is therefore valid.[10]

Part (d): We assume that we are in a situation where $(P \to Q) \wedge \sim Q$ is true. The proof can be complete by inferring that $\sim P$ must also be true (i.e., $P$ is false). Since the conjunction $(P \to Q) \wedge \sim Q$ is true, each portion $P \to Q$ and $\sim Q$ must be true (Definition 3). By contrapositive equivalence (Theorem 1 II(d)), we can infer that $\sim Q \to \sim P$ is also true. Combining this with the fact that $\sim Q$ is true, we infer (Definition 3) that the conjunction $\sim Q \wedge (\sim Q \to \sim P)$ is also true. Finally, we apply modus ponens (Part (c) of the present theorem, which we just proved) to infer that $\sim P$ is also true, as desired. □

NOTE: In our proof of part (d) when we used part (c): $P \wedge (P \to Q) \Rightarrow Q$, we actually replaced $P$ with $\sim Q$ and $Q$ with $\sim P$. Remember: Theorems 1 and 2 contain a list of logical theorems. Any of them can be used by substituting any of the logical variables by any logical variables or compound logical statements. This is what makes such theorems so useful!

## 10 Proofs and Counterexamples

New logical equivalences and implications (logical theorems) can be derived and proved using definitions, previously proved theorems, or, as a last resort, truth tables. Remember, for a logical expression (involving logical variables) to be logically equivalent to or to logically imply another, the corresponding biconditional/implication must be true for any assignment of truth values to the logical variables. It takes only a single assignment of truth values of the logical variables that would render the corresponding biconditional/implication false (a **counterexample**), to show that a logical equivalence/implication is invalid.

**EXAMPLE 5:** (a) Prove or disprove the following equivalence:
$(P \vee Q) \vee \sim R \equiv [(P \vee \sim Q) \wedge R] \to P$ using truth tables.
(b) If the equivalence in Part (a) is true, give another proof of it based on Theorem 1.
(c) Prove or disprove the following equivalence: $(P \vee Q) \wedge (\sim P \vee R) \equiv Q \vee R$.

SOLUTION: (a) A truth table is constructed below with the relevant compound statements being highlighted. Since the truth values are identical, the equivalence is established.

| $P$ | $Q$ | $R$ | $P \vee Q$ | $(P \vee Q) \vee \sim R$ | $P \vee \sim Q$ | $(P \vee \sim Q) \wedge R$ | $[(P \vee \sim Q) \wedge R] \to P$ |
|---|---|---|---|---|---|---|---|

[10] A more efficient framework to complete this proof would have been by using the method of "proof by contradiction," which will be introduced shortly.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| T | T | T | T | T | T | T | T |
| T | T | F | T | T | T | F | T |
| T | F | T | T | T | T | T | T |
| T | F | F | T | T | T | F | T |
| F | T | T | T | T | F | F | T |
| F | T | F | T | T | F | F | T |
| F | F | T | F | F | T | T | F |
| F | F | F | F | T | T | F | T |

(b) We can prove the equivalence of Part (a) directly using equivalences from Theorem 1. Below is such a proof. It certainly takes more creativity to come up with such a proof than was needed with the rote proof done in Part (a), but such an effort is rewarded with a deeper understanding of the whole theory.

$[(P \vee \sim Q) \wedge R] \to P \equiv \sim [(P \vee \sim Q) \wedge R] \vee P$ (implication as disjunction: Theorem 1 II(a))

$\equiv [(\sim P \wedge Q) \vee \sim R] \vee P$ (both of De Morgan's laws: Theorem 1 I(d))

$\equiv [(\sim P \wedge Q) \vee P] \vee \sim R$ (associativity and commutativity: Theorem 1 I(a)(b))

$\equiv [(\sim P \vee P) \wedge (Q \vee P)] \vee \sim R$ (distributivity: Theorem 1 I(c))

$\equiv [\mathrm{T} \wedge (Q \vee P)] \vee \sim R$ (tautology: Theorem 1 III(a))

$\equiv (Q \vee P) \vee \sim R$ (identity law: Theorem 1 I(g)) □

ADVICE: The reader is strongly recommended to take some time to carefully convince themselves of each line of the above argument. More general mathematical proofs are accomplished using a chain of logical inferences. If one link in the chain is wrong, then the whole proof is invalid. The situation is similar to computer programs; each line of code is logical and if the logic is flawed in just one line, the whole program is compromised. As with computer programs, the first step to learn how to write your own proofs is to be able to carefully go over and understand correct proofs written by others. Writing proofs is a bit like poetry, creativity and care are both needed. Also, like poetry, some correct proofs are more elegant than others.

(c) $(P \vee Q) \wedge (\sim P \vee R) \equiv Q \vee R$ is readily seen to be invalid; for example, if $P$ and $Q$ are false, but $R$ is true, the left and right sides have opposite truth values. The strategy we used was to try to make the left and right sides have different truth values by adjusting the truth values of the logical variables. Once again, this could have been done by a truth table (stopping once a counterexample is found), but since there are four logical variables, a truth table would have 16 rows! The implication $(P \vee Q) \wedge (\sim P \vee R) \Rightarrow Q \vee R$ is true, as the reader can easily verify.

Proving or disproving implications or equivalences can be used to assess the validity of written arguments, and this was a main application of logic by the ancient philosophers. One very common type of written argument (an inference) can be represented symbolically as follows:

Assume that $\mathsf{A}_1, \mathsf{A}_2, \cdots, \mathsf{A}_k$ are all true
Therefore ($\therefore$) $\mathsf{B}$ is true.

The validity of such an argument can be viewed as the validity of the implication:

$$A_1 \wedge A_2 \wedge \cdots \wedge A_k \Rightarrow B \qquad (1)$$

(i.e., $(A_1 \wedge A_2 \wedge \cdots \wedge A_k) \rightarrow B$ is a tautology.)[11] Logical implications (e.g., from Theorem 2) and equivalences (e.g., from Theorem 1) can be used to infer the validity of an argument, after the various basic statements are put into symbols. The following example will illustrate this method.

**EXAMPLE 6:** Determine whether each of the following arguments is valid:
(a) If Joey is a bungee jumper, then Joey is a fun person. Joey is not a bungee jumper. Therefore, Joey is not a fun person.
(b) If it rains today, Mary will go to a movie. If Mary goes to a movie, she will eat popcorn. Mary did not eat popcorn today. Therefore, it did not rain today.
(c) If Mike goes to the party, then he will either dance with Jane or not play air hockey with Chuck. If he dances with Jane, then he will come home late. If he does not play air hockey with Chuck, then he will not play cards either. Therefore, if Mike goes to the party, then either he will come home late, or he will not play cards.

SOLUTION: In each part we introduce symbols in the obvious fashion. Let us explain this abstraction just for the first statement of Part (a). We let $B$ represent the statement: "Joey is a bungee jumper," and $F$ represent the statement: "Joey is a fun person."[12] Thus "If Joey is a bungee jumper, then Joey is a fun person" is represented symbolically as $B \rightarrow F$.

Part (a): In symbols the argument becomes: $B \rightarrow F$ and $\sim B \quad \therefore \ \sim F$. Or, writing the "and" therefore symbol using logical notation: $[(B \rightarrow F) \wedge \sim B] \rightarrow \ \sim F$. If Joey, who we know is not a bungee jumper (i.e., $B$ is false) were to indeed be a fun person (i.e., $F$ is true), then both hypotheses would be satisfied ($B \rightarrow F$ becomes False $\rightarrow$ True—recall implications are always true when the hypothesis is false, and $\sim B$ is True), but the conclusion is not satisfied ($\sim F$ is false), so the whole implication becomes: True $\rightarrow$ False, which is false, so the argument is not a valid one. Invalid arguments are also called **fallacies**.

Part (b): The argument can be symbolized as follows:

$$\text{(i) } R \rightarrow M, \ \text{ (ii) } M \rightarrow P, \ \text{(iii)} \sim P \qquad \therefore \ \sim R$$

(We have numbered each of the hypotheses to facilitate the proof that follows.)
By hypothetical syllogism (Theorem 2(e)), (i) and (ii) imply (iv) $R \rightarrow P$. Next we can use (iii) and (iv) with modus tollens (Theorem 2(d)) to conclude $\sim R$, and we have thus proved the validity of the argument. In summary: we used logical inference and the truth of the three hypotheses to derive the truth of the conclusion.

Part (c): The argument can be symbolized as follows:

$$\text{(i) } P \rightarrow (D \vee \sim H), \ \text{ (ii) } D \rightarrow L, \ \text{(iii)} \sim H \rightarrow \sim C \quad \therefore \ P \rightarrow (L \vee \sim C)$$

---

[11] Any disjunction (or conjunction) of multiple statements can be represented using notation similar to the familiar sigma notation convention; for example, $\bigvee_{k=1}^{n} P_k \equiv P_1 \vee P_2 \vee \cdots \vee P_n$.

[12] For obvious reasons it is not a good idea to use either of the letters T or F to represent a statement; just for this example, however, we gave into the temptation.

*Proof #1:* We can apply the constructive dilemma implication (of Theorem 2(g)) to hypotheses (ii) and (iii) to obtain (iv) $(D \vee \sim H) \to (L \vee \sim C)$. We next apply hypothetical syllogism (Theorem 2(e)) with (i) and (iv) to obtain the desired conclusion $P \to (L \vee \sim C)$. □

*Proof #2:* Since we are trying to prove an implication $P \to (L \vee \sim C)$, we may assume also that (iv) $P$ (is true), and must show that the conclusion $L \vee \sim C$ is true. A general strategy for proving a disjunction is to assume that one of the two options is false (this gives an additional assumption to work with) and then proceed to prove the other. So we assume that $L$ is false, i.e., (v) $\sim L$ (is true). We need to show $\sim C$ is true (i.e., $C$ is false).
Using (ii) and (v) with modus tollens (Theorem 2(d)) we obtain (vi) $\sim D$.
Using (i) and (iv) with modus ponens (Theorem 2(c)) gives us (vii) $D \vee \sim H$.

*Case 1:* $D$ is true. In this case we could apply (ii) with modus ponens to conclude $L$ is true, but we know from (v) $L$ is false, so this case cannot occur.
*Case 2:* $\sim H$ is true. Now we could use (iii) and modus ponens to conclude $\sim C$, as desired. □

We make a few comments regarding the two proofs given in part (c). Proof #1 was shorter and more elegant but the second proof was more natural and it also illustrates the benefits of separating into cases (which provide additional assumptions to work with). It only used parts of Theorem 2 that are easy to remember. Honestly, most students (and professors) would not remember the constructive dilemmras from Theorem 2 two years after taking (or teaching) this course. But things like modus ponens/tollens will stay with you forever!

It is an unfortunate fact of life that (logical) fallacies do in fact occur—in conversations, in more formal scholarly arguments (both with students and academics), and even in published work and speeches. Understanding logical inference will allow you to write bullet-proof arguments and also to detect logical flaws, when they are present, in arguments of others. The above proofs demonstrate the usefulness of being familiar with a few basic theorems in logic. With experience and practice, you will begin to develop an intuition that will aid you in making an initial guess as to whether a given argument is valid. After all, trying to prove an invalid argument will be impossible, as will attempting to find a counterexample for a valid one. More complicated arguments and mathematical proofs can take more time to create or analyze. Admittedly, it is sometimes difficult to decide in which direction one should aim (i.e., either attempt to prove it, or look for a counterexample that will prove it false); but such problems are a fine instance of the art of scientific research and discovery. Truth tables can always be used to prove/disprove any purely logical assertion; in Part (c) of the preceding example, a (hand-drawn) truth table for the corresponding implication would have required 32 lines!!

In practice—both with logic and more general proofs, it is always easier to prove an equivalence by doing separate proofs of the two corresponding implications. In proving a logical implication, both of Theorem 1 and Theorom 2 may be used.

**EXAMPLE 7:** Prove the validity of the following argument.
If the Lakers win both this week's and next week's games, then they will be in the playoffs. As long as the Lakers stay injury-free, then they will win this week's games. However, if the Lakers have an injury or they lose next week's games, then the Heat will be in the playoffs. The Lakers do stay injury-free, and the Heat don't make it to the playoffs. Also, the Lakers win next week's game. Therefore, the Lakers will play in the playoffs.

SOLUTION: Using logical notation, the argument takes on the following form:

$$\text{(i) } W \wedge N \to P, \ \text{(ii)} \sim I \to W, \ \text{(iii) } (I \vee \sim N) \to H, \text{(iv)} \sim I, \text{(v)} \sim H \quad \therefore \quad P$$

From (ii) and (iv) modus ponens (Theorem 2 (c)) gives us (vi) $W$. From (iii) and (v), modus tollens gives us (vii) $\sim(I \vee \sim N)$, which by De Morgan's law (Theorem 1 I(d)) and double negation (Theorem 1 I(e)) is equivalent to (viii) $\sim I \wedge N$. By subtraction (Theorem 2 (b)) (viii) produces (ix) N, and when this is combined with (vi) and modus ponens is used with (i) we obtain the desired conclusion $P$.

Each of the proofs used in the preceding examples (for the valid arguments) was a so-called **direct proof** of the implication (1) $\mathsf{A}_1 \wedge \mathsf{A}_2 \wedge \cdots \wedge \mathsf{A}_k \Rightarrow \mathsf{B}$: we assumed the hypothesis and inferred the conclusion. It is also possible to prove an implication using an indirect proof, which is basically a proof of the (logically equivalent) contrapositive. In this format, we assume the conclusion ($\mathsf{B}$) is false, and attempt to show that one of the hypotheses ($\mathsf{A}_i$) is also false. All this really boils down to is that we are proving instead the contrapositive of (1):

$$\sim \mathsf{B} \Rightarrow \sim(\mathsf{A}_1 \wedge \mathsf{A}_2 \wedge \cdots \wedge \mathsf{A}_k)$$

(Since De Morgan's law tells us that $\sim(\mathsf{A}_1 \wedge \mathsf{A}_2 \wedge \cdots \wedge \mathsf{A}_k) \equiv \sim \mathsf{A}_1 \vee \sim \mathsf{A}_2 \vee \cdots \vee \sim \mathsf{A}_k$, we may show that the contrapositive is true by showing that at least one $\mathsf{A}_i$ is false, when $\mathsf{B}$ is false.)[13]

A particularly useful strategy for proving any implication $\mathsf{A} \Rightarrow \mathsf{B}$ is known as a **proof by contradiction**. This method works as follows: we assume both $\mathsf{A}$ and $\sim \mathsf{B}$ are true (this is the only way that the implication $\mathsf{A} \Rightarrow \mathsf{B}$ could fail to be true) and we proceed to derive a contradiction. Once a contradiction is logically derived, the proof is complete, since it shows that the simultaneous truth of both $\mathsf{A}$ and $\sim \mathsf{B}$ is not possible. This method is particularly popular for students who might get stuck on a difficult (direct) proof (where they are only assuming $\mathsf{A}$) by providing another assumption ($\sim \mathsf{B}$) with which to work.[14]

**EXAMPLE 8:** Use the method of proof by contradiction to establish the following implication:

$$[P \to (Q \vee R)] \wedge [\sim R \to P] \wedge \sim Q \Rightarrow R$$

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[13] There is one missing detail here: De Morgan's law $\sim(P \wedge Q) \equiv \sim P \vee \sim Q$ "the negation of a conjunction is the disjunction of the negations" actually works for any finite number of logical variables, although the law was stated for just two variables. The proof is rather straightforward, but requires mathematical induction, which we will introduce later on. (Readers familiar with mathematical induction should be able to easily write out the proof now, and are encouraged to do so.) At present, we can write out a proof for any (fixed) finite number of logical variables. For example, here is a proof of the statement with three logical variables: (it uses the corresponding two-variable De Morgan's law twice, along with associativity):

$$\sim(P \wedge Q \wedge R) \equiv \sim([P \wedge Q] \wedge R) \equiv \sim[P \wedge Q] \vee \sim Q \equiv [\sim P \vee \sim Q] \vee \sim Q \equiv \sim P \vee \sim Q \vee \sim R.$$

[14] In his influential textbook [Roy-88], with which a significant portion of several generations of mathematicians has been trained in their initial graduate analysis courses, Halsey Royden discourages students from resorting to proofs by contradiction, In his words: "*All students are enjoined in the strongest possible terms to eschew proofs by contradiction! There are two reasons for this prohibition: First such proofs are very often fallacious… Second, even when correct, such a proof gives little insight into the connection between* $\mathsf{A}$ *and* $\mathsf{B}$*, whereas both direct and indirect proofs construct a chain of argument connecting* $\mathsf{A}$ *and* $\mathsf{B}$." By being careful, you can certainly avoid being a reason for his first objection. His second objection does indeed highlight a good point; however, there are some cases where a proof by contradiction is really the best method to use.

SOLUTION: To use the method of proof by contradiction, in addition to the hypotheses (i) $P \to (Q \vee R)$, (ii) $\sim R \to P$, and (iii) $\sim Q$, we are also allowed to assume the negation of the conclusion: (iv) $\sim R$. Modus ponens using (ii) and (iv) gives us (v) $P$. Using modus ponens once again with (i) and (v) produces (vi) $Q \vee R$. Next using commutativity and disjunctive syllogism with (iii) and (vi), we obtain (vii) $R$, which contradicts (iv). The proof is now complete.

## 11 Logical Puzzles

Logical arguments have been popularized into puzzles, many of which make excellent problems for students of logic to practice their newly learned skills. Two particularly famous logic puzzle authors are Raymond Smullyan and Martin Gardner. Smullyan taught introductory courses in logic at Indiana University for many years and his entire course was based on logical puzzles! We close this section with two examples illustrating these sorts of puzzles; more will be given as exercises. Interested readers can find many more such problems on the Internet, in magazine columns, or in books of **logical puzzles**. Logic puzzles frequently appear even on NPR radio shows. Our first example involves a fictitious island where all of the inhabitants are either liars (they always lie) or truth tellers (they always tell the truth); such liar puzzles actually date back to the ancient Greek philosophers.

**EXAMPLE 9:** Suppose that you are a tourist on an island where all inhabitants are either liars or truth tellers. You meet three inhabitants walking together. After you greet them,

A says: "We are all liars."
B says: "No, A is the only liar."
C says: "The other two are both liars."

Can you determine, who, if any, of these three is telling you the truth?

SOLUTION: It is difficult to try and absorb all this at once, so considering individual statements and separate cases will be a big help. We will run through the different possibilities and check for consistency. A could not be a truth teller, for then his statement would be false; therefore A has to be a liar. If B were telling the truth, it would mean that C would also have to be a truth teller, but then C's statement would be false (his statement implies that B is a liar). Therefore, B is also a liar. From this we now know that C's statement is true, so he must be a (and the only) truth teller.

The next example is a famous one for which we present two slightly different proofs. The first version is a bit less formal and can be understood by people without any background in formal logic. The reader is strongly encouraged to try to answer both questions before peeking at the solution.

**EXAMPLE 10:** (*A Logical Puzzle*) Suppose that you are a prisoner on an island where every resident is either a truth teller or a liar. Truth tellers always tell the truth while liars always lie. There are two doors (a left one and a right one) and one guard. You know that one of the doors will lead to your freedom but the other will lead to your immediate death, and the guard knows which one is which.

(a) If you were allowed to ask the guard a single yes–no question, what should you ask to help you find the door that would lead to your freedom?

(b) Suppose instead that you were allowed to give the guard one statement and ask him to give you its truth value. What statement would you give the guard to help you to find freedom?

SOLUTION: Part (a): Here is one (very elegant) solution:

*If I were to ask you whether the left door would lead to my freedom, what would you say?*

That this question will work is simply based on the double negation rule (Theorem 1 I(e)). Let's analyze how this question would play out by separating into two cases:

*Case 1: The guard is a truth teller:* This is easy: "Yes" would mean you take the left door, "no" would mean you should take the right door.

*Case 2: The guard is a liar:* If the left door would lead to your freedom, and you simply asked the guard if it did, he would give you a "no" answer since he is a liar. The question being asked, however, is a bit different. It asks him to tell you what his response would be if you were to simply ask him the former question. Since he would answer "no" to the former question, in order to lie about the actual question being asked, he would need to say "yes." By the same reasoning, a "no" answer would have to mean that the right door would lead to your freedom.

Thus, with this question, it is immaterial whether the guard is a liar or a truth teller, a yes will always mean the left door is the one you should take and a no would mean you should take the right door!

Part (b): There are many possibilities; we give only one: Consider the following two basic statements:

$P$ = You are a liar, and $Q$ = The left door leads to freedom.

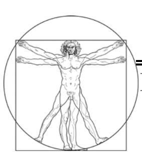
If you ask the guard the truth value of the following statement: $P \oplus Q$, in words:

*True or False? You are a liar or the left door leads to freedom, but not both.*

A true response will always mean the left door leads to freedom, while a false response will mean the right door does. We leave it to the reader to run through the four cases: guard is a truth teller or not, left door leads to freedom or not, to see that this will indeed do the job.

**EXAMPLE 11:** Suppose that you are told of a small town in which there is a barber who shaves every man and only those men who do not shave themselves. Is this possible? (Assume that every man needs shaving—at least once in a while.)

SOLUTION: Answer: No. In formal wording, the statement tells us that the barber shaves a man if and only if that man does not shave himself. *Case 1: The barber does not shave himself.* By the statement it would follow that the barber does indeed shave himself—a contradiction. *Case 2: The barber shaves himself.* The statement would again contradict this. In either case we are led to a contradiction. Therefore, such a situation is not possible.



## EXERCISES:

1. For each item below, indicate whether it is a statement. For those that are statements, attempt to determine the truth value.
   (a) Marilyn Monroe was born before John F. Kennedy.
   (b) Who assassinated John F. Kennedy?
   (c) $3^8 > 2^9$.
   (d) $x > y^2 - 7$.

(e) $x > y^2 - 7$, given that $x = 10$ and $y = 5$.
(f) Ulysses Grant was the 15th president of the United States.

2. For each item below, indicate whether it is a statement. For those that are statements, attempt to determine the truth value.
(a) Please take off your shoes.
(b) The capital of Italy is Florence.
(c) There are infinitely many *prime numbers*.
**Note:** Recall that a **prime number** (or **prime**) is an integer greater than one whose only positive integer divisors are 1 and itself. The first few primes are: 2, 3, 5, 7, 11, 13, 17.
(d) If $x^2 = 25$, then $x = 5$.
(e) If $x$ is a positive real number with $x^2 = 25$, then $x = 5$.
(f) (*Prime Pairs Conjecture*)[15] There are infinitely many **prime pairs**, that is, pairs of prime numbers that have exactly one integer between (e.g., 3, 5; 5, 7; 11, 13; 17, 19; 29, 31).

3. Determine the truth value for each of the following compound statements.
(a) The United States has 52 states or Washington DC is not a state.
(b) The United States has 52 states and Washington DC is not a state.
(c) If the United States has 52 states, then Washington DC is not a state.
(d) If Washington DC is not a state, then the United States has 52 states.
(e) The United States has 52 states, if, and only if Washington DC is not a state.

4. Determine the truth value for each of the following compound statements.
(a) Paris is the capital of France or Florence is the capital of Italy.
(b) Paris is the capital of France and Florence is the capital of Italy.
(c) If Paris is the capital of France, then Florence is the capital of Italy.
(d) If Florence is the capital of Italy, then Paris is the capital of France.
(e) Paris is the capital of France, if, and only if Florence is the capital of Italy.

5. Determine the truth value for each of the following compound statements. Assume throughout that $x = 2$, $y = -2$, and $z = 10$.
(a) $x^2 - 5y^3 > z^2$ or $z/(x^2 + y^2) < x$.
(b) $x^2 - 5y^3 > z^2$ and $z/(x^2 + y^2) < x$.
(c) If $x^2 - 5y^3 > z^2$, then $z/(x^2 + y^2) < x$.
(d) If $z/(x^2 + y^2) < x$, then $x^2 - 5y^3 > z^2$.
(e) $x^2 - 5y^3 > z^2$, if, and only if $z/(x^2 + y^2) < x$.

6. Determine the truth value for each of the following compound statements. Assume throughout that $x = 3$, $y = 2$, and $z = -5$.
(a) $x^2 - 5y^3 > z^2$ or $z/(x^2 + y^2) < x$.
(b) $x^2 - 5y^3 > z^2$ and $z/(x^2 + y^2) < x$.
(c) If $x^2 - 5y^3 > z^2$, then $z/(x^2 + y^2) < x$.

[15] The *prime pairs conjecture*, also known as the *twin primes conjecture*, is a very natural conjecture that has been around for centuries, although its exact origin is unknown. As of the writing of this book, the conjecture remains unproved. Its simplicity has helped to make resolving this conjecture (proving or disproving it) a famous problem in mathematics.

(d) If $z/(x^2+y^2)<x$, then $x^2-5y^3>z^2$.
(e) $x^2-5y^3>z^2$, if, and only if $z/(x^2+y^2)<x$.

7. Suppose that we have five cards, each of which has a positive integer (from $\{1,2,3,\cdots\}$) on one of its sides, and a letter (from $\{A,B,\cdots,Z\}$) on the other side. Suppose that the cards have been laid out on a table and they show (from left to right):

Card #1: K, Card #2: 13, Card #3: A, Card #4: 6, Card #5: X

For each statement below indicate which cards would need to be turned over to determine the truth value of the statement:
(a) If the letter on a card is a vowel (A, E, I, O, or U), then the number on the other side is greater than 10.
(b) If the letter on a card is a vowel (A, E, I, O, or U), then the number on the other side is less than 20.
(c) If the number on the card is even, then the letter on the other side must be a consonant (not a vowel).
(d) If the letter on a card is a consonant, then the number on the other side is even.
(e) If the letter on a card is a consonant, then the number on the other side is odd.
(f) For the number on a card to be odd, it is necessary that the letter on the opposite side is not a vowel.
(g) A letter on a card is even if, and only if the letter on the other side is a vowel.

8. Repeat the directions and each part of Exercise 7 if (a new set of) cards are now showing:

Card #1: 2, Card #2: 24, Card #3: E, Card #4: Z, Card #5: 21

9. Write each implication below in the form: "if … then …" and then put into proper English the converse and the contrapositive.
(a) We will go to a movie only if it rains.
(b) I will go to the party if Yumi will go.
(c) Only if I can beat Norris this weekend, I will enter the tournament.
(d) In order for Tom to make the team, it is necessary for him to be able to run a mile in under six minutes.
(e) An attractive job offer will be sufficient for Carol to move to France.
(f) In order for an infinite series $\sum_{n=1}^{\infty} a_n$ to converge, it is necessary that the terms $a_n$ tend to zero as $n$ tends to infinity.[16]

10. Write each implication below in the form: "if … then …" and then put into proper English the converse and the contrapositive.
(a) Luis will play football if it is not raining.
(b) I will go to the party only if Yumi is going.
(c) Being able to dunk a basketball is sufficient for being able to join the team.
(d) That he is on the team implies that he can dunk a basketball.
(e) An attractive job offer is necessary for Carol to move to France.
(f) If a prime number $p$ is a factor of a product of two integers $ab$, then $p$ is a factor of $a$, or $p$ is a factor of $b$.

[16] This important (albeit basic) theorem from calculus often gets (logically) misinterpreted.

11. Create truth tables for each of the following compound statements. Identify any tautologies or contradictions.
   (a) $\sim P \to P$
   (b) $P \wedge \sim (P \to P)$
   (c) $(P \wedge Q) \to P$
   (d) $P \vee (Q \to P)$
   (e) $((P \to Q) \leftrightarrow P) \to \sim Q)$
   (f) $((P \leftrightarrow Q) \wedge P) \oplus Q$
   (g) $(P \oplus Q) \to (Q \oplus R)$
   (h) $P \to (Q \leftrightarrow R)$
   (i) $(P \to Q) \to (Q \vee (R \leftrightarrow \sim P))$
   (j) $[P \to (R \wedge Q)] \leftrightarrow [\sim P \to (\sim R \vee Q)]$

12. Create truth tables for each of the following compound statements. Identify any tautologies or contradictions.
   (a) $\sim P \leftrightarrow P$
   (b) $\sim P \vee \sim (P \leftrightarrow P)$
   (c) $(P \to Q) \to (Q \to P)$
   (d) $(P \leftrightarrow Q) \to (Q \vee P)$
   (e) $(P \leftrightarrow \sim Q) \oplus (P \leftrightarrow Q)$
   (f) $((P \to Q) \to P) \to \sim Q$
   (g) $(P \to (P \wedge Q)) \to (P \wedge Q \wedge R)$
   (h) $P \to [(Q \vee P) \oplus (P \wedge R)]$
   (i) $[(P \to \sim Q) \to (Q \leftrightarrow R)] \to \sim Q$
   (j) $[R \to (P \wedge Q)] \leftrightarrow [\sim P \vee (R \to \sim Q)]$

13. (a) A statement of the form "*P* unless *Q*" is formally equivalent to "*P* or *Q*," however, there is sometimes ambiguity in spoken language where at times the exclusive or is intended. In formal language, which of the following implications is thus equivalent to "P unless Q?"

   (i) $P \to Q$ (ii) $P \to \sim Q$ (iii) $\sim P \to Q$ (iv) $\sim P \to \sim Q$

   (b) Express the statement: "I will go to the movies unless Diane calls" in the form: "if …, then …". Then form the converse and contrapositive both in the "if …, then …" form and using the word "unless."

14. (a) See Exercise 13, and then determine which of the following implications the statement "*P* unless *Q*" is equivalent to:

   (i) $Q \to P$ (ii) $Q \to \sim P$ (iii) $\sim Q \to P$ (iv) $\sim Q \to \sim P$

   (b) Express the statement: "I will spend spring break in Hawaii unless Professor Garnett schedules the midterm the day after the break." in the form: "… only if …". Then form the converse and contrapositive both in the "… only if …" form and using the word "unless."

15. (a) Use truth tables (one for the left compound statement and another for the right) to establish De Morgan's law: (Theorem 1 Part I(d)) $\sim (P \vee Q) \equiv \sim P \wedge \sim Q$.

   (b) Give another proof of this De Morgan law that is based on directly analyzing and comparing the definitions of the relevant logical operators. See, for example, the proof of Part (c) of Theorem 2 and the solution of Example 3 (where the result of Theorem 1 Part II(a) is derived).

16. Repeat both parts of Exercise 15 for De Morgan's Law: $\sim (P \wedge Q) \equiv \sim P \vee \sim Q$.

17. Repeat both parts of Exercise 15 for each of the following equivalences from Theorem 1.
   (i) (*Commutativity of Conjunction*) $P \wedge Q \equiv Q \wedge P$.
   (ii) (*Associativity of Conjunction*) $(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$.
   (iii) (*Double Negation*) $\sim (\sim P) \equiv P$.

(iv) (*Exportation*) $P \to (Q \to R) \equiv (P \wedge Q) \to R$.

18. Repeat both parts of Exercise 15 for each of the following equivalences from Theorem 1.
(i) (*Commutativity of Disjunction*) $P \vee Q \equiv Q \vee P$.
(ii) (*Associativity of Disjunction*) $(P \vee Q) \vee R \equiv P \vee (Q \vee R)$.
(iii) (*Absorption*) $P \vee (P \wedge Q) \equiv P$.
(iv) (*Biconditional as Implication*) $P \leftrightarrow Q \equiv (P \to Q) \wedge (Q \to P)$.

19. (a) Use truth tables to establish the hypothetical syllogism: (Theorem 2(e)) $(P \to Q) \wedge (Q \to R) \Rightarrow P \to R$.
(b) Give another proof of the implication of Part (a) using any earlier implication of Theorem 2 (Parts (a) through (d)) and/or any of the equivalences of Theorem 1.

20. Prove each of the following implications using any legitimate method.
(a) $(P \to Q) \wedge (R \to S) \Rightarrow [(P \vee R) \to (Q \vee S)]$.
(b) $(P \to Q) \wedge (R \to S) \Rightarrow [(P \wedge R) \to (Q \wedge S)]$.

21. Repeat both parts of Exercise 19 for the disjunctive syllogism (Theorem 2(f)): $(P \vee Q) \wedge \sim P \Rightarrow Q$.

22. Determine whether the following is a tautology: $(P \to Q) \wedge (\sim Q \to R) \to (P \to R)$.

23. Establish the following logical equivalences by using only Theorem 1:
(a) $P \leftrightarrow Q \Leftrightarrow (P \wedge Q) \vee (\sim P \wedge \sim Q)$.
(b) $(P \vee Q) \vee \sim R \Leftrightarrow [(P \vee \sim Q) \wedge R] \to P$.

24. Express the negations of each of the implications in Exercise 9 in English and without the use of any implication.

25. A radio advertisement states: "If you don't come in to Johnson Toyota this weekend to buy your new Toyota, you will be paying too much for your new car." This exercise will logically analyze the actual and intended meanings of this statement. We let $P$ = "You don't come in to Johnson Toyota this weekend" and $Q$ = "You will be paying too much for your new car." The statement can thus be represented as $P \to Q$.
(a) Suppose that you did come in to Johnson Toyota, bought a new car and subsequently found out that the competitor Anderson Toyota had been selling the exact same model that you bought for $2,000 less. Would Johnson Toyota's advertised claim be contradicted, i.e., constitute false advertising? Do you think that Johnson Toyota had intended (and radio listeners had interpreted) the possibility of such an occurrence?
(b) Give examples of situations for the remaining three rows of the truth value for the Johnson Toyota conditional, and compare the intended truth of the statement with the corresponding logical truth value in each case.
(c) Reword Johnson Toyota's statement into a logically coherent form that correctly conveys their intended (and interpreted) message. How often do you encounter such wording in radio, television, and newspapers?

26. (*A Logical Puzzle*) A car driver was listening to a discussion of the World Cup results on a radio talk show. He heard one speaker say: "Either Italy was first, or Germany was second, but not both." During the next statement, some static cut off one of the words, and all the driver heard

was: "Either France was second, or Germany was <static>, but not both." The radio host then said that with this information it was possible to determine the ranking (the host did not know about the static that this driver heard). Nevertheless, the driver was able to determine the top three rankings from what he heard. Which countries came in first, second, and third?
**Suggestion:** Separate into the three possible cases for which ranking could have been said under the static. Only one of these three cases leads to a unique permissible ranking.

27. (*A Logical Puzzle*) Three friends, one from France, one from Germany, and one from Italy are driving together when they hear the following comment on the radio about the recent World Cup matches: "The top three countries were Italy, France, and Germany. Either Italy outranked France, or Germany came in first, but not both." After hearing that, the Frenchman said that even though he knew the standing of his country's team, it was not possible for him to determine the ranking. The Italian, who did not listen to what the Frenchman just said, also said it was not possible for him to determine the ranking, even though he knew how his country's team placed. The German, who heard everything but knew nothing about the matches, was able to determine the top three ranking. Which countries came in first, second, and third?
**Suggestion:** The radio announcer's comment reduces the number of possible rankings to three. Write them out, then use the Italian and Frenchman's comments to rule out two of these three rankings.

28. (*A Logical Puzzle*) You are a tourist visiting an island where all inhabitants are either liars or truth tellers. You run into two inhabitants who tell you: A: "B would say I lie." B: "This is true." What conclusions can you draw about the types of persons A or B are (liar or truth teller)?

29. (*Logical Puzzles*) Assume that you are visiting an island where all inhabitants are either truth tellers—who always tell the truth, or liars—who always lie. Assume that all the inhabitants know one another and their truth status: whether they are truth tellers or liars. Suppose that you run into some inhabitants and hear the following conversations. For each part determine whether it is possible to determine each of their truth status. Explain your answers.
(a) A: "We are both truth tellers."
B: "A is a liar."
(b) A: "I love you B."
B: "I love you A."
A: "You are a liar B, but I still love you."

30. (*Logical Puzzles*) Assume that you are visiting an island where all inhabitants are either truth tellers—who always tell the truth, or liars—who always lie. Assume that all the inhabitants know one another and their truth status: whether they are truth tellers or liars. Suppose that you run into some inhabitants and hear the following conversations. For each part determine whether it is possible to determine each of their truth status. Explain your answers.
(a) A: "C and D are both truth tellers.",
B: "C is a truth teller, but D is a liar."
C: "Neither B nor D is a liar."
D: "Neither A nor B is a liar."
(b) A: We are all truth tellers.
B: We are all liars.
C: They are both liars.

31. (*A Logical Puzzle*) In Example 10, suppose instead that there were three doors: 1, 2, and 3, of which only one would lead to you (the prisoner) to freedom, while the other two would lead to your death. The guard knows which is which, but you don't know if he is a liar or a truth teller.

(a) Explain why you would not be able to decide the correct door if you were allowed to ask the guard only one true/false question.
(b) Give a strategy for determining the desired door if you were allowed to ask the guard two true/false questions.

32. (*A Logical Puzzle*) Four suspects were questioned in a bank robbery, and each made the following two statements: A: "I did not do it, B did." B: "A did not do it. C did." C: "B did not do it, I did." D: C did not do it, A did." A polygraph machine indicated that each person told the truth in just one of the two statements. From this information, is it possible to determine the bank robber? Explain your answer.

NOTE: Exercises 31–35 deal with the concept of a *functionally complete* set of operators. A set of logical operators is called **functionally complete** if any compound statement (involving any of the full set of logical operators introduced in this section: $\{\wedge,\vee,\sim,\rightarrow,\leftrightarrow\}$) can be expressed using only the operators in this set, and perhaps also any number of pairs of parentheses. Certainly the full set of operators $\{\wedge,\vee,\sim,\rightarrow,\leftrightarrow\}$ is a (trivial) example of a functionally complete set of operators. Since any implication can be expressed using only the disjunction and negation operators (Theorem 1 II(a)), it follows that we can remove the implication and still have a functionally complete set of operators: $\{\wedge,\vee,\sim,\leftrightarrow\}$.

33. (*A Functionally Complete Set of Operators*) (a) Show that $\{\wedge,\vee,\sim\}$ is a functionally complete set of operators.
(b) (*Disjunctive Normal Form*) Given any compound statement with $n$ logical variables: $P_1, P_2, \cdots, P_n$, the corresponding truth table will have $2^n$ rows in it. For each row of this truth table for which the statement is true, the truth values of the variables will be a certain length $n$ sequence of T's and F's. Form the conjunction $\mathsf{A}$ of $Q_1, Q_2, \cdots, Q_n$, where each $Q_i$ equals either $P_i$ or $\sim P_i$ according as to whether the truth value of $P_i$ (in this particular row of the truth value) is T or F, respectively. For example, if $n = 4$, and the truth values in a row (where the compound statement is true) are T, F, F, T, then $\mathsf{A}$ would be $P_1 \wedge \sim P_2 \wedge \sim P_3 \wedge P_4$. Observe that $\mathsf{A}$ is true only for this single row of the truth table, i.e., when $P_1 = T, P_2 = F$, $P_3 = F$, and $P_4 = T$, and false for all other rows in the truth table. This construction gets repeated, and another such $\mathsf{A}$ gets created for each row in the truth table for which the original statement is true. The disjunction of all of these $\mathsf{A}$ 's will do the job. This representation of $\mathsf{A}$ is called the *disjunctive normal form*.
Find the disjunctive normal form of each of the following compound statements:
(i) $P \rightarrow Q$ (ii) $P \vee (Q \rightarrow P)$ (iii) $P \rightarrow (Q \leftrightarrow R)$

34. (*A Functionally Complete Sets of Operators*) (a) Show that $\{\wedge,\sim\}$ is a functionally complete set of operators.
(b) Show that $\{\vee,\sim\}$ is a functionally complete set of operators.
**Suggestion:** Use the result of Exercise 33 in conjunction with De Morgan's laws.

35. (*Non Functionally Complete Sets of Operators*) (a) Show that $\{\wedge\}$ is not a functionally complete set of operators.
(b) Show that $\{\vee\}$ is not a functionally complete set of operators.
(c) Show that $\{\vee,\wedge\}$ is not a functionally complete set of operators.
**Suggestion:** Examine (compound) statements with just one logical variable.

36. (*A Single Logical Operator Can Be a Functionally Complete Set of Operators*) We define the **nor** operator on two logical variables $P$ and $Q$ as follows:

    $P$ nor $Q$ (denoted $P \downarrow Q$ ) is true only when $P$ and $Q$ are both false, and is false in all other cases.

    Thus, the nor operator is just the negation of or (nor = not or), i.e., $P \downarrow Q \equiv \sim (P \vee Q)$.

    This exercise will outline a proof that $\{\downarrow\}$ constitutes a functionally complete set of operators. This is surprising in light of Exercise 35.

    (a) Show that $\sim P \equiv P \downarrow P$.

    (b) Show that $P \vee Q \equiv (P \downarrow Q) \downarrow (P \downarrow Q)$.

    (c) Use the result of a previous exercise to deduce that $\{\downarrow\}$ is a functionally complete set of operators.

37. (*A Single Logical Operator Can Be a Functionally Complete Set of Operators*) The negation of the conjunction operator is called the **nand** (not and) operator; we denote this operator as: $P \uparrow Q$. Thus, $P \uparrow Q \equiv \sim (P \wedge Q)$. Show that $\{\uparrow\}$ is a functionally complete set of operators.

    **Suggestion:** Try to develop an argument similar to that outlined in Exercise 36 for the nor operator.

# CHAPTER 1 ANSWERS AND SOLUTIONS:

**#1.** (a) False statement (MM 1926, JFK 1917) (b) Not a statement (c) True statement (d) Not a statement (e) False statement (f) False statement (Grant was the 18th president)
**#3.** (a) True (b) False (c) True (d) False (e) False
**#5.** (a) F or T = True (b) F and T = False (c) If F, then T = True (d) If T, then F = False (e) F if, and only if T = False
**#7.** (a) 3, 4 (b) 3 (c) 3, 4 (d) 1, 2, 5 (e) 1, 4, 5 (f) 2, 3 (g) 1, 2, 3, 4, 5
**#9.** (a) If we go to a movie, then it will rain. Converse: If it will rain, then we will go to a movie. Contrapositive: If it will not rain, then we will not go to a movie. (b) If Yumi is going to the party, then I will go. Converse: If I go to the party, then Yumi will come. Contrapositive: If I do not go to the party, then Yumi will not go. (c) If I enter the tournament, then I can beat Norris this weekend. Converse: If I can beat Norris this weekend, then I will enter the tournament. Contrapositive: If I cannot beat Norris this weekend, then I will not enter the tournament. (d) If Tom makes the team, then he is able to run a mile in under six minutes. Converse: If Tom is able to run a mile in under six minutes, then he makes the team. Contrapostive: If Tom cannot run under a six minute mile, then he will not make the team. (e) If Carol gets an attractive job offer, then she will move to France. Converse: If Carol moves to France, then she got an attractive job offer. Contrapositive: If Carol does not move to France, then she did not get an attractive job offer. (f) If $\sum_{n=1}^{\infty} a_n$ converges, then the terms $a_n$ tend to zero as $n$ tends to infinity. Converse: If the terms $a_n$ tend to zero as $n$ tends to infinity, then $\sum_{n=1}^{\infty} a_n$ converges. Contrapositive: If $\sum_{n=1}^{\infty} a_n$ diverges (or does not converge), then the terms $a_n$ do not tend to zero as $n$ tends to infinity.

**#11.** (a)

| $P$ | $\sim P \to P$ | | |
|---|---|---|---|
| T | F | T | |
| F | T | F | |

(b)

| $P$ | $P \wedge \sim (P \to P)$ | | |
|---|---|---|---|
| T | F | F | T |
| F | F | F | T |

(c) A tautology

| $P$ | $Q$ | $(P \wedge Q) \to P$ | |
|---|---|---|---|
| T | T | T | T |
| T | F | F | T |
| F | T | F | T |
| F | F | F | T |

(d)

| $P$ | $Q$ | $P \vee (Q \to P)$ | |
|---|---|---|---|
| T | T | T | T |
| T | F | T | T |
| F | T | F | F |
| F | F | T | T |

(e)

| $P$ | $Q$ | $((P \to Q) \leftrightarrow P) \to \sim Q)$ | | | |
|---|---|---|---|---|---|
| T | T | T | T | F | F |
| T | F | F | F | T | T |
| F | T | T | F | T | F |
| F | F | T | F | T | T |

(f)

| $P$ | $Q$ | $((P \leftrightarrow Q) \wedge P) \oplus Q$ | | |
|---|---|---|---|---|
| T | T | T | T | F |
| T | F | F | F | F |
| F | T | F | F | T |
| F | F | T | F | F |

(g)

| $P$ | $Q$ | $R$ | $(P \oplus Q) \to (Q \oplus R)$ | | |
|---|---|---|---|---|---|
| T | T | T | F | T | F |
| T | T | F | F | T | T |
| T | F | T | T | T | T |
| T | F | F | T | F | F |
| F | T | T | T | F | F |
| F | T | F | T | T | T |
| F | F | T | F | T | T |
| F | F | F | F | T | F |

(h)

| $P$ | $Q$ | $R$ | $P \to (Q \leftrightarrow R)$ | |
|---|---|---|---|---|
| T | T | T | T | T |
| T | T | F | F | F |
| T | F | T | F | F |
| T | F | F | T | T |
| F | T | T | T | T |
| F | T | F | T | F |
| F | F | T | T | F |
| F | F | F | T | T |

(i)

| $P$ | $Q$ | $R$ | $(P \to Q) \to (Q \vee (R \leftrightarrow \sim P))$ | | | | |
|---|---|---|---|---|---|---|---|
| T | T | T | T | T | T | F | F |
| T | T | F | T | T | T | T | F |
| T | F | T | F | T | F | F | F |
| T | F | F | F | T | T | T | F |
| F | T | T | T | T | T | T | T |
| F | T | F | T | T | T | F | T |
| F | F | T | T | T | T | T | T |
| F | F | F | T | F | F | F | T |

(j)

| $P$ | $Q$ | $R$ | $[P \to (R \wedge Q)] \leftrightarrow [\sim P \to (\sim R \vee Q$ | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| T | T | T | T | T | T | F | T | F | T |
| T | T | F | F | F | F | F | T | T | T |
| T | F | T | F | F | F | F | T | F | F |
| T | F | F | F | F | F | F | T | T | T |
| F | T | T | T | T | T | T | T | F | T |
| F | T | F | T | F | T | T | T | T | T |
| F | F | T | T | F | F | T | F | F | F |
| F | F | F | T | F | T | T | T | T | T |

**#13.** (a) $P$ unless $Q \equiv \; \sim P \to Q$. (b) If I do not go to the movies, then Diane will have called. Converse: If Diane calls, then I will not go to the movies. $\equiv$ Diane does not call unless I don't go to the movies. (This form is certainly awkward.) Contrapositive: If Diane does not call, then I will go to the movies. $\equiv$ Diane will have called unless I went to the movies. (Note: Do not worry to much about the form of the verb used in these constructions; there is some flexibility here since the logic does not concern itself with verb tenses.) **#15.** (a) The needed truth table is shown below:

| $P$ | $Q$ | $P \vee Q$ | $\sim (P \vee Q)$ | $\sim P$ | $\sim Q$ | $\sim P \wedge \sim Q$ |
|---|---|---|---|---|---|---|
| T | T | T | F | F | F | F |
| T | F | T | F | F | T | F |
| F | T | T | F | T | F | F |
| F | F | F | T | T | T | T |

(b) The conjunction $\sim P \wedge \sim Q$ will be true (in one out of the four cases) when both $\sim P$ is true and $\sim Q$ is true. In other words, when both $P$ and $Q$ are false. Since any conjunction (of two logical variables) is true in three out of four cases, its negation will be true in exactly one out of four cases. With both $P$ and $Q$ false, the statement $\sim (P \vee Q)$ becomes $\sim (\mathrm{F} \vee \mathrm{F}) \equiv \sim \mathrm{F} \equiv \mathrm{T}$, thus $\sim (P \vee Q)$ has the same truth values as $\sim P \wedge \sim Q$.

**#17.** (i) (a) The one-step truth table of $Q \wedge P$ is readily seen to be identical to that for $P \wedge Q$ (Table 2). (b) Alternatively, any conjunction is true exactly when both parts are true, for either $P \wedge Q$ or $Q \wedge P$ this amounts to $P$ = T and $Q$ = T, so the truth tables are identical. (ii) (a) The truth table shown below establishes the equivalence.

| $P$ | $Q$ | $R$ | $P \wedge Q$ | $(P \wedge Q) \wedge R$ | $Q \wedge R$ | $P \wedge (Q \wedge R)$ |
|---|---|---|---|---|---|---|
| T | T | T | T | T | T | T |
| T | T | F | T | F | F | F |
| T | F | T | F | F | F | F |
| T | F | F | F | F | F | F |
| F | T | T | F | F | T | F |
| F | T | F | F | F | F | F |
| F | F | T | F | F | F | F |
| F | F | F | F | F | F | F |

(b) Alternatively, using the fact that a conjunction of two statements is true precisely when each of the statements is true, we see that $(P \wedge Q) \wedge R = \mathrm{T}$ exactly when $P \wedge Q = \mathrm{T}$ and $R$ = T. Using this fact again we get that $P \wedge Q = \mathrm{T}$ exactly when $P$ = T and $Q$ = T. In summary, the only row of the truth table for $(P \wedge Q) \wedge R$ that results in a true truth value is when all of $P$, $Q$, and $R$ = T. A nearly identical argument shows that this is the only row of the truth table for $P \wedge (Q \wedge R)$ that makes this latter statement true.

(iii) (a) The simple truth table for the double negation rule shown below establishes the equivalence $\sim(\sim P) \equiv P$.

| $P$ | $\sim P$ | $\sim(\sim P)$ |
|---|---|---|
| T | F | T |
| F | T | F |

(b) Alternatively, $\sim(\sim P) = \mathrm{T}$ exactly when $\sim P = \mathrm{F}$, which happens exactly when P = T, so $\sim(\sim P) \equiv P$.

(iv) (a) A truth table that establishes the exportation rule is shown below:

| $P$ | $Q$ | $R$ | $Q \to R$ | $P \to (Q \to R)$ | $P \wedge Q$ | $(P \wedge Q) \to R$ |
|---|---|---|---|---|---|---|
| T | T | T | T | T | T | T |
| T | T | F | F | F | T | F |
| T | F | T | T | T | F | T |
| T | F | F | T | T | F | T |
| F | T | T | T | T | F | T |
| F | T | F | F | T | F | T |
| F | F | T | T | T | F | T |
| F | F | F | T | T | F | T |

(b) Alternatively, working with the fact that implications are true "most of the time" (in three out of four cases), we separately determine what truth values make each of the two statements false. First, $P \to (Q \to R)$ = F, if and only if $P$ = T and $Q \to R = \mathrm{F}$. The latter equation forces $Q$ = T and $R$ = F. Similarly, if we "solve the logical equation" $(P \wedge Q) \to R = \mathrm{F}$, we get that $P \wedge Q = \mathrm{T}$ and $R$ = F. The former forces $P = Q$ = T. Thus the values of $P$, $Q$, and $R$ that render $P \to (Q \to R)$ false are the same as those that render $(P \wedge Q) \to R$ false, so the equivalence is proved.

**#19.** (a) The required truth table is shown below:

| $P$ | $Q$ | $R$ | $P \to Q$ | $Q \to R$ | $(P \to Q) \wedge (Q \to R)$ | $P \to R$ |
|---|---|---|---|---|---|---|
| T | T | T | T | T | T | T |
| T | T | F | T | F | F | F |
| T | F | T | F | T | F | T |
| T | F | F | F | T | F | F |
| F | T | T | T | T | T | T |
| F | T | F | T | F | F | T |
| F | F | T | T | T | T | T |
| F | F | F | T | T | T | T |

To establish the implication $(P \to Q) \wedge (Q \to R) \Rightarrow P \to R$, we need only look at the rows of this truth table where the hypotheses (first part) $(P \to Q) \wedge (Q \to R)$ is true (these are the darker shaded rows), and check that the corresponding values of the conclusion (second part) $P \to R$ are also true. Since this is indeed the case, the implication is proved. (b) Assume that the hypothesis (i) $(P \to Q) \wedge (Q \to R)$ is true. We must prove that the conclusion $P \to R$ is also true. So assume that (ii) $P$ is true. It suffices to show that $R$ is also true (i.e., we assume $P$ and must deduce $R$). By subtraction (Theorem 2 (b)) from (i) we get (iii) $P \to Q$. From (ii) and (iii) and modus ponens (Theorem 2 (c)) we deduce (iv) $Q$. Another application of subtraction to (i) gives us (v) $Q \to R$, [17] which, when combined with (iv), another application of modus ponens gives $R$, as desired.

**#21.** (a) The truth table needed to establish the implication $(P \vee Q) \wedge \sim P \Rightarrow Q$, is shown on the below. We need only check that when the hypothesis $(P \vee Q) \wedge \sim P$ is true, so is the conclusion $Q$. This is indeed the case (indicated by the shaded row of the truth table).

| $P$ | $Q$ | $P \vee Q$ | $\sim P$ | $(P \vee Q) \wedge \sim P$ | $Q$ |
|---|---|---|---|---|---|
| T | T | T | F | F | T |
| T | F | T | F | F | F |
| F | T | T | T | T | T |
| F | F | F | T | F | F |

(b) Here is an analytic proof: We assume that the hypothesis is true: (i) $(P \vee Q) \wedge \sim P$. Our task is to deduce $Q$ (i.e., the truth of the conclusion). From (i) and subtraction (Theorem 2(b)) we deduce (ii) $P \vee Q$ and (iii) $\sim P$. We may rewrite the disjunction (ii) as an implication (iv) $\sim P \to Q$ (Theorem 1 II(a)). Applying modus ponens (Theorem 2 (c)) with (iii) and (iv) we deduce $Q$, as desired.

**#23.** (In both proofs we use the names of each equivalence as given in Theorem 1.)

(a)

$P \leftrightarrow Q \equiv (P \to Q) \wedge (Q \to P)$ (Biconditional as Implications)

$\equiv (\sim P \vee Q) \wedge (\sim Q \vee P)$ (Implication as Disjunction)

$\equiv [\sim P \wedge (\sim Q \vee P)] \vee [Q \wedge (\sim Q \vee P)]$ (Distributivity)

$\equiv [(\sim P \wedge \sim Q) \vee (\sim P \wedge P)] \vee [(Q \wedge \sim Q) \vee (Q \wedge P)]$ (Distributivity)

$\equiv [(\sim P \wedge \sim Q) \vee \mathbf{F}] \vee [\mathbf{F} \vee (Q \wedge P)]$ (Contradictions)

$\equiv (\sim P \wedge \sim Q) \vee (Q \wedge P)$ (Identity Laws)

$\equiv (P \wedge Q) \vee (\sim P \wedge \sim Q)$ (Commutativity) □

[17] Technically, we also used commutativity (Theorem 1(a)) $P \wedge Q \equiv Q \wedge P$, in order to deduce $Q$ from the subtraction implication $(P \wedge Q \Rightarrow P)$. In the future, we sometimes will use commutativity, associativity (Theorem 1(b)), and double negation (Theorem 1(e)) without explicit mention.

(b)

$(P \vee Q) \vee \sim R$

$\equiv [\mathbf{T} \wedge (P \vee Q)] \vee \sim R$ (Identity Laws)

$\equiv [(\sim P \vee P) \wedge (Q \vee P)] \vee \sim R$ (Tautology, Commutativity)

$\equiv [(\sim P \wedge Q) \vee P] \vee \sim R$ (Distributivity)

$\equiv [(\sim P \wedge Q) \vee \sim R] \vee P$ (Commutativity)

$\equiv [\sim (P \vee \sim Q) \vee \sim R] \vee P$ (De Morgan′s law)

$\equiv \sim [(P \vee \sim Q) \wedge R] \vee P$ (De Morgan′s law)

$\equiv [(P \vee \sim Q) \wedge R] \rightarrow P$ (Implication as Disjunction) □

**#25.** (a) This situation gives F $\rightarrow$ T, which is a true implication. The company probably did not intend for this to happen, but is does not logically contradict their claim. (b) The implication T $\rightarrow$ T would correspond to someone going to another dealer and paying more (too much) for a new Toyota. The logical truth of this statement corresponds to the intended truth of the claim. The implication T $\rightarrow$ F would correspond to someone going to another dealer and not paying more (too much) for a new Toyota. Logically, this implication is false and so would contradict the advertised claim. Finally, the implication F $\rightarrow$ F would correspond to someone coming (and buying) a car from Johnson Toyota and not paying too much (i.e., more than at other dealers). The logical truth of this statement agrees with the intended truth. (c) A phrase such as: "our prices are the lowest," would work. Such phrases are common (but not worth much without some sort of guarantee).

**#27.** Italy = 1$^{st}$, France = 2$^{nd}$, Germany = 3$^{rd}$

**#29.** (a) A is a liar, B is a truth teller.

(b) A: I love you B., B: I love you A., A: You are a liar B, but I still love you. Let's go through each of the four possible truth status cases, so see which, if any, are consistent:

Case 1: A TT, B L: OK

Case 2: A TT, B TT: No! (A's second statement would be false.)

Case 3: A L, B L: OK (Since the first two statements would be lies, A's second statement would be: T ^ F = F.)

Case 4: A L, B TT: OK

So the truth status of each of A,B cannot be determined.

**#31.** (a) The two possible answers could not distinguish between three doors. (b) A simple strategy would be to use the idea of Part (b) of the solution of Example 1.6. First point to the first door and ask the guard: True or False? You are a liar or this door leads to freedom, but not both. As in the example, a true answer will always mean this door leads to freedom. If the guard answers false, then you know the freedom door must be one of the other two, so you can easily modify the Example 1.6 solution question to distinguish between the two remaining doors.

**#35.** (a) Since $P \vee P \equiv P$, any grouping involving parentheses, a single logical variable $P$, and the $\vee$ symbol will always be equivalent (by repeatedly using the stated equivalence) to $P$. In particular, it will not be possible to represent the statement $\sim P$. (b) Repeated use of the identities $P \vee P \equiv P$ and $P \wedge P \equiv P$, allows us to reduce any logical statement built up with the single logical variable $P$, along with the operators $\wedge$ and/or $\vee$, along with balanced parentheses to prove such a statement is logically equivalent to $P$. It is therefore not possible to represent the statement $\sim P$ in this way.

**#37.** *Sketch of proof:* First show that $P \uparrow P \equiv \sim P$ and $(P \uparrow P) \uparrow (Q \uparrow Q) \equiv P \vee Q$, and then use De Morgan's law (or Exercise 34(b)).