Chapter 5: Counting Techniques, Combinatorics, and Generating Functions

In discrete mathematics problems, one often needs to know how many (or approximately how many) objects belong to a certain set. Such counting problems may be interesting in their own right, or constitute an integral part of the solution to another problem. Many important and difficult problems in probability (Chapter 6) amount to counting problems. The counting techniques that we learn in this chapter will also be useful in assessing and comparing the speeds of algorithms, where it is required to get rough estimates of how many logical/arithmetical operations need to be performed in the execution of an algorithm (usually as a function of the input size). This latter application of counting methods is known as the *complexity theory of algorithms*, and will be studied in Chapter 7. After introducing an assortment of useful counting methods in Sections 5.1 and 5.2, Section 5.3 will discuss the theory of *generating functions*. Every sequence has a (unique) generating function. Generating functions are a powerful tool that often allow difficult or seemingly intractable counting problems to be translated into much simpler questions by translating a combinatorial question into a corresponding question about an appropriately formulated generating function.

5.1: FUNDAMENTAL PRINCIPLES OF COUNTING

The subject of sophisticated counting methods has evolved into an important branch of mathematics called *combinatorics*. In this and the next section we will introduce some of the central ideas of combinatorics that frequently arise in discrete structures. Since students often have difficulty remembering how and when to apply some of these methods, we will motivate several key principles by examples. Keeping a collection of such examples in mind for comparisons and contrasts will help the reader in deciding what principles are applicable in order to solve various problems. We first introduce some notation for the number of elements in a finite set.

NOTATION: If S is any finite set, the symbol $|S|$, which can be read as the **cardinality** of S, denotes the number of elements in the set S.

The Multiplication Principle

EXAMPLE 5.1: (*Motivating example for the multiplication principle*) Arlo packs three shirts, two ties, and three pairs of pants for a business trip. How many different outfits can Arlo put together during this trip? Assume that an outfit consists of one choice each of a shirt, tie, and a pair of pants, and that any differences in the choices lead to different outfits.

SOLUTION: One approach is to represent the sequence of choices by a so-called *tree diagram*;[1] such a diagram is shown in Figure 5.1. Notice that each outfit corresponds to a unique sequence of choices of a shirt (from S1, S2, and S3), tie (from T1 and T2), and pants (from P1, P2, and P3), and this in turn corresponds to a unique path down the tree, which is completely determined by where it lands on the bottom. Notice that at the end of each stage, total number of choices is the number from the previous stage multiplied by the number of choices available at the current stage. Thus we have shown that Arlo is able to put together a total of $3 \cdot 2 \cdot 3 = 18$ outfits.

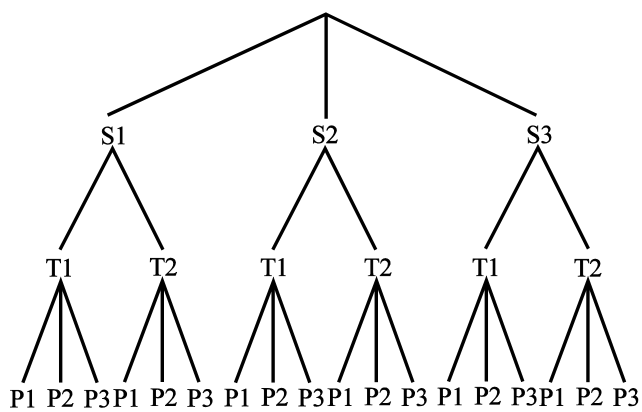

FIGURE 5.1: A tree diagram for the counting problem of Example 5.1. To put together an outfit, we start at the top (root) of the tree, and first choose one of three shirts {S1, S2, S3}, next we choose a tie from {T1, T2}, and finally we choose a pair of pants from {P1, P2, P3}. Each path from top to bottom represents a different permissible outfit, and no other outfits can be put together.

Using a set theoretic approach, the outfits can be viewed to correspond to elements in the Cartesian product $S \times T \times P$,

[1] We will give a more formal development of trees (tree diagrams) in Chapter 8. For now we treat the concept as an intuitive one.

$|S \times T \times P| = |S| \cdot |T| \cdot |P|$, we have another way to count the different outfits.

PROPOSITION 5.1: If $S_1, S_2, \cdots, S_k$ are finite sets, then the cardinality of their Cartesian product $S_1 \times S_2 \times \cdots \times S_k$ (see Section 1.3) is given by:

$$|S_1 \times S_2 \times \cdots \times S_k| = |S_1| \cdot |S_2| \cdot \cdots \cdot |S_k|.$$

Proof: We use induction on *k*.

1. *Basis Step*: The identity simply states that $|S_1| = |S_1|$.
2. *Inductive Step*: Assuming that $k \geq 1$ and the identity is true for *k* factors, we need to show it is valid for *k* + 1 factors. Write out the elements of the last factor S_{k+1} as $\{a_1, a_2, \cdots, a_N\}$, so that $N = |S_{k+1}|$. Now the elements of $S_1 \times S_2 \times \cdots \times S_k \times S_{k+1}$ fall into *N* disjoint subsets, $T_1, T_2, \cdots, T_N$, depending on their last entry (of the S_{k+1} factor). Therefore, for $1 \leq j \leq N$, we can write

$$T_j = \{(s_1, s_2, \cdots, s_k, a_j) \mid \forall_{i, 1 \leq i \leq k} [s_i \in S_i]\}.$$

The elements of each of these T_j's are thus in one-to-one correspondence with the elements of $S_1 \times S_2 \times \cdots \times S_k$, so by the inductive hypothesis, we have for each index *j*, that $|T_j| = |S_1 \times S_2 \times \cdots \times S_k| = |S_1| \cdot |S_2| \cdot \cdots \cdot |S_k|$. Since there are $N = |S_{k+1}|$ T_j's, and they are disjoint, it follows that $|S_1 \times S_2 \times \cdots \times S_k \times S_{k+1}| =$ $|T_1 \cup T_2 \cup \cdots \cup T_N| = N \cdot |S_1| \cdot |S_2| \cdot \cdots \cdot |S_k| = |S_1| \cdot |S_2| \cdot \cdots \cdot |S_{k+1}|$. □

Either of the two methods of the above example works easily to establish the following more general principle:

THE MULTIPLICATION PRINCIPLE: Suppose that a sequence of choices is to be made and that there are m_1 options for the first choice, m_2 options for the second choice, and so on, up to the *k*th choice. If these choices can be combined freely, then the total number of possible outcomes for the whole set of choices is $m_1 \cdot m_2 \cdot \cdots \cdot m_k$.

The multiplication principle is extremely useful. To apply it to a counting problem, one must be able to recast the problem at hand into a sequence of unrestricted choices. The following examples will demonstrate this technique.

EXAMPLE 5.2: A standard Hawaii license plate consists of a group of three letters followed by a group of three digits; see Figure 5.2.

(a) How many (standard) Hawaii license plates can the state produce?

(b) If on the island of Maui, the first letter of the plate must be "M," how many (standard) Maui plates can be produced?

FIGURE 5.2: A standard Hawaii license plate.

SOLUTION: Part (a): We view creating a Hawaii plate as making a sequence of six unrestricted choices; for each letter slot we have 26 choices, while for the digit slots we have 10 choices. Hence, by the multiplication principle, the total number of Hawaii plates will be $26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 17,576,000$.

Part (b): Since the first letter is already specified, making a Maui plate can be viewed as a sequence of five choices, with the total number being $26^2 \cdot 10^3 =$ 676,000.

EXAMPLE 5.3: A three-member committee is to be formed from the US Senate, which has 100 members (2 from each state). The committee will have a chairperson, a vice-chair, and a spokesperson.

(a) How many different such committees can be formed?

(b) How many if Senator A must be on it?

(c) How many if Senators B and C will serve together or not at all?

SOLUTION: Part (a): We break up the formation of the committee into the following sequence of three choices: first choose a chair (100 senators to choose from), next, after a chair has been chosen, choose a vice-chair (from the 99 senators remaining), finally, from the 98 senators remaining, we choose the spokesperson.[2] The multiplication principle tells us that there can be a total of $100 \cdot 99 \cdot 98 = 970,200$ such committees.

Part (b): We give two different methods:

Method 1: (*Separate into disjoint cases first*) We have learned early on in the last chapter that problems can often be reduced to simpler ones using cases. For the problem at hand there are three natural cases: A serves as chair, vice-chair, or spokesperson. These three cases are disjoint (no matter how the rest of the committee is formed). (Why?) Using the multiplication principle to fill the remaining slots, by disjointness, we may add up the results to get the answer to Part (b): $1 \cdot 99 \cdot 98 + 99 \cdot 1 \cdot 98 + 99 \cdot 98 \cdot 1 = 3 \cdot 99 \cdot 98 = 29{,}106$ (the factor 1 in each of the three terms represents that there is only one choice for the corresponding slot, since Senator A will occupy that slot in each case).

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[2] Of course, this method of choosing a committee has no bearing on the process of how the Senate might actually put together such a committee (usually by nominations and voting); we cast the task as a sequence of choices solely as a mathematical device to solve the counting problem.

Method 2: (*Use the multiplication principle directly*):

$$\underset{\substack{\text{places to put}\\ \text{Senator A}}}{3} \cdot \underset{\substack{\text{choices to fill}\\ \text{first remaining}\\ \text{slot}}}{99} \cdot \underset{\substack{\text{choices to fill}\\ \text{second remaining}\\ \text{slot}}}{98}$$

Part (c): Separating into the two natural cases: (i) neither B nor C serves, and (ii) both B and C serve (which give rise to disjoint sets of committees) seems like the only way to go here. Each of the two cases is amenable to the multiplication principle:

$$\underbrace{\underset{\substack{\text{choices for}\\ \text{the chair}}}{98} \cdot \underset{\substack{\text{choices for}\\ \text{the vice-chair}}}{97} \cdot \underset{\substack{\text{choices for the}\\ \text{spokesperson}}}{96}}_{\text{neither B nor C serve}} + \underbrace{\underset{\substack{\text{positions}\\ \text{for B}}}{3} \cdot \underset{\substack{\text{positions}\\ \text{for C}}}{2} \cdot \underset{\substack{\text{choices for the}\\ \text{remaining position}}}{98}}_{\text{both B and C serve}} = 913{,}164.$$

EXERCISE FOR THE READER 5.1: A professional basketball team is arranging a publicity photograph with 5 players taken from its active list of 12 players.
(a) If the players are to be lined up in a row, how many different photograph arrangements are possible?
(b) How many such arrangements are possible if players K and S refuse to appear in the lineup together?

The multiplication principle has both practical and theoretical utility. We use it next to give a proof of an important fact that was mentioned in Section 1.3 (recall that we also gave a proof of this result in Section 3.1 using mathematical induction, see Proposition 3.3).

PROPOSITION 5.2: A set *S* with a finite number *n* of elements has 2^n subsets.

Proof: We list the elements of the *S* as $\{a_1, a_2, \cdots, a_n\}$. We can view the formation of a subset $B \subseteq S$ as a sequence of *n* choices, the *i*th choice being whether to include the element a_i in the subset *B*. Since each of these *n* steps has two choices (i.e., either $a_i \in B$ or $a_i \notin B$), it follows from the multiplication principle that there are a total of $\underbrace{2 \cdot 2 \cdot \cdots \cdot 2}_{n \text{ factors}} = 2^n$ subsets of *S*. □

The Complement Principle

Another simple yet often useful rule is a consequence of the basic fact that for any subset $S \subseteq U$ (the universal set), *U* is the disjoint union of *S* and its complement $\sim S$. If *U* is a finite set, this implies that $|U| = |S| + |\sim S| \Rightarrow |S| = |U| - |\sim S|$. We reiterate this in words:

THE COMPLEMENT PRINCIPLE: Suppose the universal set is finite. The number of elements in a set equals the number of elements in the (finite) universal set, less the number of elements that are not in the set.

EXAMPLE 5.4: For security reasons, a university's finance office requires students to create a six-character password to log into their accounts. Passwords must contain at least one digit and at least one letter.
(a) How many passwords are possible if the protocol is not case-sensitive?
(b) What if the protocol is case-sensitive?

SOLUTION: Let D denote the set of all six character strings that contain at least one digit, and L the set of all six character strings that contain at least one letter. We wish to count the number of passwords in the set $D \cap L$. The sets D and L are difficult to count directly, but their complements are easy. For example, $\sim D$ is the set of all six character passwords that contain no digits, and therefore consist only of letters. By the multiplication principle, the number of such passwords is 26^6 for Part (a) and 52^6 for Part (b). In the same fashion, $|\sim L| = 10^6$ (for both Parts (a) and (b)). Also, letting S denote the (universal) set of all six character passwords, the multiplication principle gives that $|S| = 36^6$ for Part (a) and $|S| = 62^6$ for Part (b).
The complement principle and then De Morgan's law allow us to write

$$|D \cap L| = |S| - |\sim (D \cap L)| = |S| - |\sim D \cup \sim L|.$$

Now, (fortunately) the sets $\sim D$ and $\sim L$ are disjoint, so $|\sim D \cup \sim L| = |\sim D| + |\sim L|$. We now have all the information we need to answer the questions:

Part (a): $|S| - (|\sim D| + |\sim L|) = 36^6 - 26^6 - 10^6 \approx 1.86687 \times 10^9$.
Part (b): $|S| - (|\sim D| + |\sim L|) = 62^6 - 52^6 - 10^6 \approx 3.70286 \times 10^{10}$.
(Certainly either protocol should be sufficient to accommodate any university.)

The Inclusion-Exclusion Principle

When counting elements in unions of sets that are disjoint, one simply can add up the numbers of elements of the individual sets. In cases of nondisjoint sets, one needs to be more careful.

THE INCLUSION-EXCLUSION PRINCIPLE: (a) (*For two sets*) If A and B are finite sets, then $|A \cup B| = |A| + |B| - |A \cap B|$.
(b) (*For three sets*) If A, B, and C are finite sets, then
$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$.
(c) (*General case*) If $A_1, A_2, \cdots, A_n$ is a collection of finite sets, then

$$\begin{aligned} |A_1 \cup A_2 \cup \cdots \cup A_n| = & \sum_{i=1}^{n} |A_i| - \sum_{i_1 < i_2} |A_{i_1} \cap A_{i_2}| + \cdots \\ & + (-1)^{a+1} \sum_{i_1 < i_2 < \cdots < i_a} |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_a}| + \cdots \\ & + (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n| . \end{aligned} \tag{1}$$

In this identity, we start off by adding the numbers of elements in each set, then subtract the numbers of elements in each possible intersection of two of the sets, then add the numbers of elements in all possible intersections of three of the sets, and so on.

We point out that in the special case in which the sets are pairwise disjoint, i.e. $\forall i \neq j\ [A_i \cap A_j = \varnothing]$, all of the intersection cardinalities are 0s, so formula (1) simply becomes $|A_1 \cup A_2 \cup \cdots \cup A_n| = |A_1| + |A_2| + \cdots + |A_n|$. We have already used this simple formula when we broke counting arguments into disjoint cases (see the solution of Example 5.3(b), for example). We caution the reader to be extremely careful to resist using this tempting formula, unless he/she is absolutely certain that the needed pairwise disjointness requirement is satisfied.

Proof: We give different proofs for each Part (a) and (b). We prove Part (a) directly and analytically, whereas for Part (b), we use Venn diagrams and go at it sequentially. The reader might wish to supply the alternative proof for each part.

Part (a): We can express A as the disjoint union of $A \sim B$ and $A \cap B$. Consequently, $|A| = |A \sim B| + |A \cap B|$. In the same fashion, $|B| = |B \sim A| + |A \cap B|$. But $A \cup B$ is the disjoint union of the three sets $A \sim B$, $B \sim A$, and $A \cap B$. This yields $|A \cup B| = |A \sim B| + |B \sim A| + |A \cap B|$. Comparing these three equations produces the desired result.

Part (b): We use $|A| + |B| + |C|$ as our naïve "first approximation" to $|A \cup B \cup C|$. If we draw a Venn diagram to see how many times each constituent portion is counted, we arrive at the picture in Figure 5.3(a), where we put an integer in each portion to indicate how many times it was counted in $|A| + |B| + |C|$. Our goal is to have each portion counted exactly once. The two set intersection portions are all counted twice (except for the three-set portion in the middle), so as our next approximation, we subtract off the counts of all two-set intersections: $|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C|$. This second approximation gives the modified counts shown in Figure 5.3(b). All of the portions of the Venn diagram are fine, except for the central portion, corresponding to where all three sets intersect. The count for this portion is now zero. This is easy to compensate for—we simply need to add $|A \cap B \cap C|$ to get our final approximation, which is the asserted inclusion-exclusion formula for three sets. Figure 5.3(c) shows that we now have the desired counts (=1) on all portions of the Venn diagram.

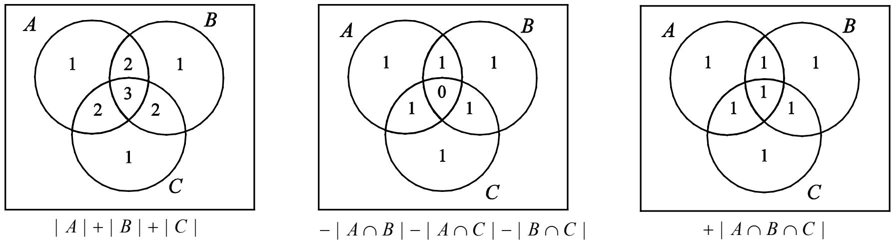

FIGURE 5.3: Proof of the inclusion-exclusion principle for three sets, in three steps. (a) (left) the initial approximation for $|A\cup B\cup C|$ counts some regions twice and one region three times, (b) (middle) the compensated approximation counts all regions once except for the central region, (c) (right) the final approximation counts all regions correctly.

Part (c): Equation (1) can be proved by mathematical induction (Exercise 31), but a more elegant proof can be given with the material developed in the next section (Exercise for the Reader 5.10). □

EXAMPLE 5.5: A university mathematics department has 75 applied mathematics majors, 50 pure mathematics majors, and 32 mathematics education majors. Of these students, there are 17 who list both pure and applied math as their majors, 13 and 11 who list math education and pure, or applied math, respectively, and finally there are five triple majors. How many math majors are there? How many of these are only majoring in applied math?

SOLUTION: The reader is encouraged to draw a Venn diagram for this problem. With the obvious notation, we are given that $|A|=75$, $|P|=50$, $|E|=32$, $|A\cap P\cap E|=5$, $|A\cap P|=17$, $|E\cap P|=13$, and $|E\cap A|=11$. The inclusion-exclusion principle for three sets now gives us the total number of math majors $|A\cup P\cup E|$ to be 75 + 50 + 32 – (17 + 13 + 11) + 5 = 121. To get the number of students who are majoring only in applied mathematics $(A\sim(P\cup E))$, we subtract from the total number 75 of applied math majors those who are double majors with pure math ($|A\cap P|-|A\cap P\cap E|$ = 17 – 5 =) 12, those who are double majors with math education ($|A\cap E|-$ $|A\cap P\cap E|$ = 11 – 5 =) 6, and the five who are triple majors ($A\cap P\cap E$). This gives us 75 – 12 – 6 – 5 = 52 students who are single majors in applied mathematics.

EXAMPLE 5.6: How many positive integers less than 2009 are divisible by none of 3, 4, or 10?

SOLUTION: We first introduce some convenient notation. For each positive integer n, we let D_n

$D_4 =$ Clearly $|D_n| = \lfloor 2008/n \rfloor$. Also, since an integer a is divisible by both n and m if, and only if a is divisible by the least common multiple of n and m (= lcm(n,m)), we may conclude that $D_n \cap D_m = D_{\text{lcm}(n,m)}$, and also $D_n \cap D_m \cap D_k = D_{\text{lcm}(n,m,k)}$.

The problem is to find $|\sim D_3 \cap \sim D_4 \cap \sim D_{10}|$. By De Morgan's law, this number is the same as $|\sim (D_3 \cup D_4 \cup D_{10})|$, and by the complement principle, this number equals 2008 − $|D_3 \cup D_4 \cup D_{10}|$. From the facts mentioned above and the inclusion-exclusion principle, we may now easily perform the needed computations. We first find the three single set counts:

$$|D_3| = \lfloor 2008/3 \rfloor = 669,\ \ |D_4| = \lfloor 2008/4 \rfloor = 502,\ |D_{10}| = \lfloor 2008/10 \rfloor = 200.$$

Next we compute the three double-set counts,

$$|D_3 \cap D_4| = |D_{12}| = \lfloor 2008/12 \rfloor = 167,\ |D_3 \cap D_{10}| = |D_{30}| = \lfloor 2008/30 \rfloor = 66,$$
$$|D_4 \cap D_{10}| = |D_{20}| = \lfloor 2008/20 \rfloor = 100,$$

and, finally, the single triple-set count:

$$|D_3 \cap D_4 \cap D_{10}| = |D_{60}| = \lfloor 2008/60 \rfloor = 33.$$

Invoking the inclusion-exclusion principle allows us now to arrive at the answer:

$$2008 - |D_3 \cup D_4 \cup D_{10}| = 2008 - (669 + 502 + 200) + (167 + 66 + 100) - 33 = 937.$$

This result can be easily verified using a simply programmed computer loop, and readers are encouraged to perform such a check.

EXERCISE FOR THE READER 5.2: How many positive integers less than 3601 are divisible by at least one of 2, 3, 5, or 11?

EXERCISE FOR THE READER 5.3: How many Hawaiian license plates (see Example 5.2) do not contain any of the strings, “CIA,” “FBI,” or “GOD?”

The Pigeonhole Principle

Our next principle, known as the *pigeonhole principle*, will probably seem so intuitively obvious that it is hardly worth mentioning. Along with its generalization, the pigeonhole principle turns out to be quite a useful tool.

THE PIGEONHOLE PRINCIPLE: If there are more than k pigeons placed into k pigeonholes, then there must be at least one pigeonhole with more than one pigeon occupying it (see Figure 5.4).

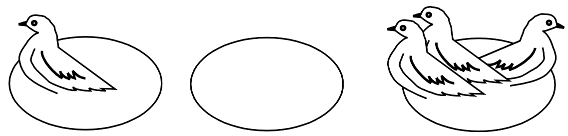

FIGURE 5.4: Illustration of the pigeonhole principle with $k = 3$ pigeonholes: if there are more pigeons (here four) roosting in the pigeonholes than there are pigeonholes, then at least one pigeonhole must have more than one pigeon.

The proof of the pigeonhole principle is an easy proof by contradiction. If every pigeonhole has at most one pigeon in it, then the total number of pigeons would have to be less than or equal to the number of pigeonholes, which is k, a contradiction to the fact that there were supposed to be more pigeons than pigeonholes.

We first give an example of some rather basic consequences of the pigeonhole principle and then proceed to give some more surprising applications.

EXAMPLE 5.7: (a) If a company has 1000 employees, it must have at least two employees who share the same birthday. This follows from the pigeonhole principle with the $k = 366$ possible birthdays being the pigeonholes, and the 1000 ($> k$) employees serving as the pigeons. (We really needed only 367 employees for this to be true.)

(b) If Joey works at Vitali's restaurant 20 evenings in March and Vivian works there 12 evenings in March then they must share at least one common evening of work. This follows from the pigeonhole principle with the pigeonholes being the $31 - 20 = 11$ days that Joey does not work and the pigeons being the 12 days that Vivian works. If there were no overlap, one of the pigeonholes would have two pigeons, i.e., this would mean that Vivian was working twice in the same evening. This is clearly impossible, so Joey and Vivian must indeed share a shift.

EXAMPLE 5.8: If seven points are randomly selected on (the circumference of) a circle of radius 1, show that at least two of these points will lie at a distance of less than 1 from each other.

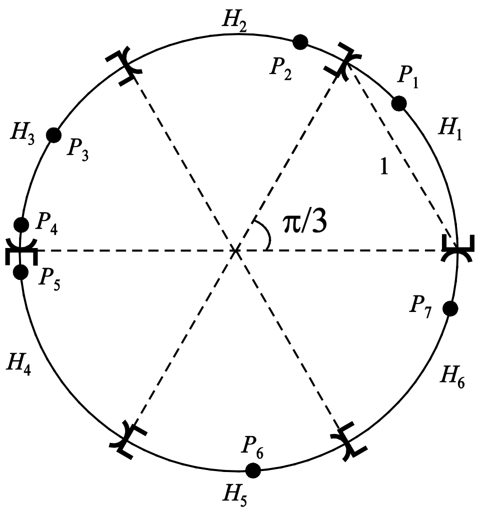

FIGURE 5.5: The six pigeonholes (H_i) and seven (random) pigeons (P_i) for Example 5.8.

SOLUTION: We partition the circle into six congruent arcs: $H_1, H_2, \cdots, H_6$ —these will serve as the pigeonholes; see Figure 5.5. As shown in the figure, each of these arcs has distance between its endpoints equal to one. But since each arc contains only one of its endpoints, it follows that any two points in such an arc H_i will be separated by a distance less than one. We let the pigeons be the seven random points on the circle. By the pigeonhole principle, at least one of the arcs must contain at least two of the randomly selected points. (In Figure 5.5, H_3 contains P_3 and P_4.)

EXERCISE FOR THE READER 5.4: (a) Show that if five points are randomly selected inside or on an equilateral triangle of side length one, then there will be two of these points whose distance between is not more than 1/2. (b) Show that if the five points are selected to be inside the triangle, then there will be two whose distance is less than 1/2.

EXERCISE FOR THE READER 5.5: Show that if we take any set of 51 integers from the set $\{1, 2, \cdots, 100\}$, then one of the integers in this set must divide some other integer in this set.

EXERCISE FOR THE READER 5.6: Suppose that n is a positive integer. Show that in any set of $n + 1$ integers none of which is divisible by n, there must exist two integers whose difference is divisible by n.

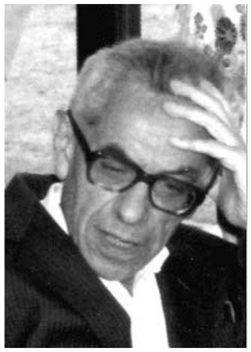

FIGURE 5.6: Paul Erdös (1913–1996), Hungarian mathematician

The following application of the pigeonhole principle comes from a paper coauthored by the illustrious Hungarian mathematician Paul Erdös,[3] see [ErSz-35].

PROPOSITION 5.3: Let n be a positive integer. Every sequence of $N = n^2 + 1$ distinct real numbers $(a_1, a_2, \cdots, a_N)$ contains a subsequence of length n + 1 that is either increasing or decreasing.

The proposition implies, for example, that if a brigade of 101 soldiers are standing in a lineup, then it is always possible to find 11 to take a step forward so that their heights will be nonincreasing or nondecreasing, as we go from left to right.[4]

Proof: We proceed by the method of contradiction. Assume that no such subsequences exist for a given sequence $(a_1, a_2, \cdots, a_N)$.

we associate an ordered pair of positive integers (I_j, D_j), where

I_j a_j, and D_j is the

length of the longest decreasing subsequence starting at a_j.

$(a_1, a_2, \cdots, a_N) = (5, 3, 7, 6, 8)$, then $I_2 = 3$ (corresponding to the

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[3] Paul Erdös (pronounced "air-dish") was born in Hungary shortly before the outbreak of World War I. His parents were both mathematics teachers. Erdös's two elder sisters had perished to scarlet fever only a few days before his birth, so his parents were particularly protective of their last child. His parents were non-practicing Jews, and this led to numerous difficulties for the family. Erdös' mathematics focused on problem solving rather than general theoretical developments, and he was one of the greatest problem solvers of all time, publishing over 1500 papers in his lifetime, mostly in the areas of combinatorics and number theory. He led a simple life that allowed him to focus almost exclusively on mathematics. Although he had been offered many permanent decent positions that his friends encouraged him to accept, he preferred to live out of his suitcase, to travel around the world, and meet other mathematicians with whom to work. Additionally he was very modest and noncompetitive. For example, he had independently discovered a very elegant proof of the *prime number theorem* with (Princeton mathematician) Atle Selberg. Although both had agreed to publish their papers back-to-back in the same journal, the latter jumped ahead and won the prestigious *Fields Medal* for it. Erdös spent very little of the money that he earned (from prizes, lectures, and temporary contracts), instead, he used it to put up prizes to encourage work on difficult problems. He had such a wide array of collaborators that the concept of a mathematician's *Erdös number* came into being. Erdös's Erdös number is 0. All of his coauthors have Erdös number 1. Others who have written a joint paper with someone with Erdös number 1 have number 2, and so on. If there is no chain of coauthorships connecting someone with Erdös, then that person's Erdös number is said to be infinite.

[4] In case some heights are the same, we (artificially) perturb them by very small numbers, so as to make them all different. For example, if we are measuring heights only to the nearest half inch, and if we had six men who were 75.5 inches tall, we would put these six numbers to be 75.5000, 75.5001, …, 75.5005. Once the proposition is applied, we could convert the heights back to their original numbers, and then still have a sequence that is nondecreasing/nonincreasing.

increasing subsequence (3,6,8)) and $D_2 = 1$ (corresponding to the decreasing subsequence (3)). These ordered pairs (I_j, D_j) will serve as our pigeonholes. In light of our (contradiction seeking) hypothesis, we must have $I_j, D_j \le n$ for every index *j*. Thus, by the multiplication principle, there are at most n^2 pigeonholes, and since we have more pigeons, the pigeonhole principle implies that there must be two different terms a_j, a_k with $j < k$, and with $I_j = I_k$ and $D_j = D_k$. We will separate into two cases to arrive at contradictions.

Case 1: $a_j < a_k$ Here we can juxtapose a_j to the beginning of an increasing subsequence of length $I_j = I_k$ that starts at a_k to get an increasing subsequence of length $I_j + 1$—a contradiction!

Case 2: $a_j > a_k$ Here we can juxtapose a_j to the beginning of a decreasing subsequence of length $D_j = D_k$ that starts at a_k to get a decreasing subsequence of length $D_j + 1$—a contradiction!

Since we have arrived at a contradiction in all of the possible cases, the proof of the theorem is complete. □

The Generalized Pigeonhole Principle

The pigeonhole principle guarantees that as long as there is at least one more pigeon than there are pigeonholes, then (at least) one pigeonhole has double occupancy. In case there are a lot more pigeons than pigeonholes, the following generalization of the pigeonhole principle allows us to draw more accurate conclusions.

THE GENERALIZED PIGEONHOLE PRINCIPLE: If there are *N* pigeons placed into *k* pigeonholes, then there must be at least one pigeonhole with at least $\lceil N/k \rceil$ pigeons occupying it.

Proof: Just as for the first version of the pigeonhole principle, we will proceed by the method of contradiction. Suppose that all of the pigeonholes contained fewer than $\lceil N/k \rceil$ pigeons. Then the total number of pigeons contained in all of the pigeonholes could be at most $k \cdot (\lceil N/k \rceil - 1)$, but this number is less than $k \cdot ([N/k + 1] - 1) = N$, which is a contradiction. □

EXAMPLE 5.9: If we apply the generalized pigeonhole principle to example 5.7(a), (the company that had 1000 employees), we would arrive at the stronger conclusion that there must be at least 3 ($= \lceil 1000/366 \rceil$) employees who share the same birthday. If there were more than 1097 employees, we could say that at least four employees share the same birthday.

EXERCISES 5.1:

1. A standard California license plate consists of a single digit, followed by three letters, followed by three digits.
   (a) How many standard California license plates can be made?
   (b) How many can be made if no letters or digits can be used twice?

2. (a) A menu special at a restaurant offers three courses: appetizer, main course, and dessert. For the appetizer, one can choose either house salad, Caesar salad, or soup of the day. For the main course, the choices are either prime rib, chicken Marengo, or sautéed shrimp. The dessert choices are chocolate mousse, mixed fresh fruit platter, or ice cream cake. How many different menus can be created if one must choose one item from each course?
   (b) Repeat Part (a) with an additional cheese course with choices of either: camembert, garlic herb cheese or goat cheese is added along with a beverage choice of beer, iced tea, cola, or milk.

3. A man is deciding among four restaurants: Italian, Thai, Chinese, or steakhouse, and then four after-dinner activities: a movie, dancing, bowling, or a basketball game on which to take his date. How many different dates can he put together if he includes both dinner and one after-dinner activity?

4. In a trip from San Diego to Seattle, suppose that we are considering three routes from San Diego to Los Angeles, four routes from Los Angeles to San Francisco, and five routes from San Francisco to Seattle. How many different trips could we plan from San Diego to Seattle that go through Los Angeles and San Francisco?

5. A restaurant manager is trying to assign five workers, Andy, Beth, Charlie, Doris, and Earl to five different jobs for the evening.
   (a) If all of these workers can do any of these jobs, how many job assignments are possible?
   (b) How about if Andy, Beth, and Earl cannot do the first two jobs?
   (c) How about if Andy, Beth, and Earl cannot do the last job?

6. How many positive integers are less than 8000 that have no repeated digits and no occurrences of the digits 2, 4, or 8?

7. An international student club has 12 members: 3 Chinese, 2 Vietnamese, 1 French, 3 Germans, 2 Japanese, and 1 Australian.
   (a) The group elects a president, vice president, and treasurer. In how many ways can this be done?
   (b) Same question as (a) but with the requirement that at least one of the elected be Chinese.
   (c) Same question as (a) but with the restriction that the Germans and Japanese refuse to serve together.
   (d) Same question as (a) but with the requirement that two members A and B will either serve together or not at all.

8. (a) How many functions are there from $\{1, 2, 3, 4, 5\}$ to $\{1, 2, 3, 4, 5, 6, 7\}$?
   (b) How many of the functions f in Part (a) satisfy $f(i) \in \{1, 2\}$ for $i = 1, 2, 3$?
   (c) How many of the functions f in Part (a) are one-to-one functions?

9. (a) How many functions are there from $\{1, 2, 3, 4, 5, 6, 7\}$ to $\{1, 2, 3, 4, 5\}$?
   (b) How many of the functions f in Part (a) satisfy $f(i) \in \{1, 2\}$ for $i = 1, 2, 3$?
   (c) How many of the functions f in Part (a) are one-to-one functions?

10. Suppose that n and m are positive integers.
    (a) How many functions are there from $\{1, 2, \ldots, n\}$ to $\{1, 2, \ldots, m\}$?

(b) How many of the functions f in Part (a) satisfy $f(i) \in \{1,2\}$ for $i = 1, 2, 3$? (Assume that $n > 2$ and $m > 1$.)
(c) How many of the functions f in Part (a) are one-to-one functions?

11. A restaurant's lunch menu has three courses:
Cheeses: Brie, Jarlsberg, smoked hickory, or Swiss
Salads: Caesar, romaine, tossed greens, or chicken
Sandwiches: BLT, tuna, turkey, or Italian
(a) If one is allowed to choose exactly one item from exactly two of the three different courses, how many selections would be possible?
(b) If a couple is allowed to choose exactly two items from exactly two of the three different courses, how many selections would be possible?

12. A password protocol for a certain network requires that all passwords use digits or lower-case letters and consist of six to eight characters.
(a) How many passwords are possible?
(b) How many passwords are possible that include at least one digit and at least one letter?
(c) How many passwords are possible that include at least two letters?
(d) How many passwords are possible that contain no vowels?
(e) How many passwords are possible that include at least one digit and at least one consonant?

13. A password protocol for a certain network requires that all passwords use digits or letters and consist of five to seven characters. The protocol is case sensitive.
(a) How many passwords are possible?
(b) How many passwords are possible that include at least one digit and at least one letter?
(c) How many passwords are possible that include at least one digit, at least one lower-case letter, and at least one upper-case letter?
(d) How many passwords are there that contain the string "CAT" (in any mixture of cases)?

14. In a certain state, of the 500 largest companies, 200 offer (free) health insurance (to all employees), 300 offer dental insurance, and 150 offer life insurance. Moreover, 150 offer both health and dental insurance, 100 offer health and life, 75 offer dental and life, and 50 offer all three types of coverage. How many companies offer none of these three coverages?

15. (a) Use the inclusion-exclusion principle to determine the number of positive integers less than 6000 that are divisible by at least one of the primes 3, 5, or 7.
(b) Use the inclusion-exclusion principle to determine the number of positive integers less than 6000 that are divisible by none of the primes 3, 7, or 11.
(c) Write and execute computer loops that will check your answers to (a) and (b).

16. (a) Use the inclusion-exclusion principle to determine the number of positive integers less than 4000 that are divisible by at least one of the numbers 4, 6, or 10.
(b) Use the inclusion-exclusion principle to determine the number of positive integers less than 4000 that are divisible by none of the numbers 5, 6, or 15.
(c) Write and execute computer loops that will check your answers to (a) and (b).

17. (a) Use the inclusion-exclusion principle to determine the number of positive integers less than 6000 that are divisible by at least one of the primes 3, 5, 7, or 11.
(b) Write and execute a computer loop that will check your answers to (a).

18. Use (1) to write down an explicit formula for the inclusion-exclusion principle for five sets.

19. (a) Use the inclusion-exclusion principle to determine the number of positive integers less than 4000 that are divisible by at least one of the numbers 4, 6, 10, 15, or 20.
(b) Use the inclusion-exclusion principle to determine the number of positive integers less than 4000 that are divisible by none of the numbers 2, 3, 5, 7, or 11.
(c) Write and execute computer loops that will check your answers to (a) and (b).

20. (a) Explain why among a group of 60 foreign exchange students from the United States, at least 2 came from the same state.
(b) What is the minimum number of cards that must be drawn from a shuffled standard deck of 52 cards to guarantee that there will be at least one pair? Provide an example to show that if one less than this number is drawn, a pair need not come up.

21. (a) Explain why in a group of 21 men whose heights range from 5 feet to 6 feet 7 inches, there must be at least two whose height, rounded to the nearest inch, must be the same.
(b) What is the minimum number of cards that must be drawn from a shuffled standard deck of 52 cards to guarantee that there will be at least three cards of the same suit? Provide an example to show that if one less than this number is drawn, three same suit cards need not appear.

22. Show that if 13 points are chosen on a circle of radius 1, then at least two of these points will be within a distance of 1/2 from one another.

23. (a) Show that if 10 points are randomly selected in the interior of an equilateral triangle of side length 1, then there will be two of these points whose distance from one another is less than 1/3.
(b) Show the result of Part (a) is sharp by constructing an example of nine points within an equilateral triangle of side length 1 such that the distance between any pair is greater than or equal to 1/3.

24. (a) Show that if five points are randomly selected within the interior of a square of side length 2, then there will be two of these points whose distance from one another is less than $\sqrt{2}$.
(b) Show that if nine points are randomly selected with the interior of a square of side length 2, then there will be three of these points such that the distance between any pair is less than $\sqrt{2}$.

25. Given a positive integer $n > 1$, determine the minimum positive integer K_n, such that if any K_n points are selected in the interior of an equilateral triangle of side length 1, then there must be at least two of these points that lie at a distance less than $1/n$ from each other.

26. Prove that given any 11 positive integers, at least two of them will have their difference being divisible by 10.

27. Prove that given any seven positive integers, at least two of them will have either their sum or their difference being divisible by 10.
Suggestion: Use the pigeonhole principle with the pigeonholes determined by the last digit of each integer in such a way that two integers in the same pigeonhole will have either their sum or their difference divisible by 10.

28. Suppose that we have a list of 8 positive integers (with possible duplications) that add up to 20. Show that we can always draw a sublist (with possible duplications) from this list whose elements add up to four.
Suggestion: First show that the list must contain 1 or 2, then use the pigeonhole principle.

29. At a medium-sized university there are 869 students taking an introductory statistics course this semester among 10 sections. What is the smallest possible enrollment in the largest section?

30. How many truth tables are possible for logical statements containing n logical variables?

31. Use mathematical induction to prove the inclusion-exclusion principle (1) for finite unions of finite sets:

$$\begin{aligned}|A_1 \cup A_2 \cup \cdots \cup A_n| &= \sum_{i=1}^{n} |A_i| - \sum_{i_1 < i_2} |A_{i_1} \cap A_{i_2}| + \cdots \\ &\quad + (-1)^{a+1} \sum_{i_1 < i_2 < \cdots < i_a} |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_a}| + \cdots \\ &\quad + (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n| .\end{aligned}$$

5.2: PERMUTATIONS, COMBINATIONS, AND THE BINOMIAL THEOREM

The Difference Between a Permutation and a Combination

Our next example will compare and contrast the concepts of a *permutation* and a *combination*. Although the example is small enough to count by brute-force, we will solve it with an approach that lends itself easily to generalization.

EXAMPLE 5.10: (*Motivating example for understanding the difference between a permutation and a combination*) Suppose that Mr. Vitali has interviewed four women: Alice, Betty, Christine, and Daisy, to fill three job openings at his Italian restaurant, and that all turned out to be equally qualified.
(a) In how many ways can Mr. Vitali hire a cashier, a cook, and a waitress from these four applicants?
(b) In how many ways can Mr. Vitali hire three of these four women to work as waitresses?

SOLUTION: Although the two questions are similar, there is one very important, yet perhaps subtle, difference. In the first question, the order in which Mr. Vitali hires/assigns the women is definitely important, since the jobs are all different. In the second question, however, order/assignment is not relevant, since the women are being hired for identical positions.

Part (a): (***Order matters: permutations***) We have already shown how to answer questions like this using the multiplication principle; the answer is given by:

$$\underset{\substack{\text{women to}\\ \text{hire as}\\ \text{cashier}}}{4} \cdot \underset{\substack{\text{women left}\\ \text{to hire as}\\ \text{cook}}}{3} \cdot \underset{\substack{\text{women left}\\ \text{to hire as}\\ \text{waitress}}}{2} = 24.$$

Each of these 24 outcomes can be viewed as an ordered triple, e.g., (B, C, D), and is called a *permutation (or rearrangement) of the four objects* A, B, C, D, *taken three at a time*. The obvious abbreviations are being used, e.g., the triple (B, C, D) would correspond to hiring B(etty) as the cashier, C(hristine) as the cook, and D(aisy) as the waitress.

Part (b): (***Order doesn't matter: combinations***) Consider a typical permutation of the 24 outcomes for Part (a), e.g., (B, C, D). If we *rearrange* (or *permute*) the women in this list, we get a different outcome/permutation for Part (a), e.g., (C, D, B), would represent the outcome of hiring C as the cashier, D as the cook, and B as the waitress. Thus different **permutations** correspond to different outcomes for Part (a). However, for Part (b), all of these permutations of the list (B, C, D) would correspond to the same outcome for Part (b), since all of B, C, D would be hired as waitresses. Thus, for Part (b), this outcome should really be represented as the set {B, C, D}, since order does not matter (in set notation). Such an object is called a *combination of the four objects* A, B, C, D, *taken three at a time*.

In order to arrive at the answer to Part (b), we first find out how many different permutations there are of (just) the ordered list (B, C, D). The multiplication principle can easily give us the answer:

$$\underset{\substack{\text{choose one}\\ \text{of B,C,D}\\ \text{for 1st slot}}}{3} \cdot \underset{\substack{\text{choose one}\\ \text{remaining two}\\ \text{for 2nd slot}}}{2} \cdot \underset{\substack{\text{only one}\\ \text{choice for}\\ \text{last slot}}}{1} = 6.$$

This example is (intentionally) small enough so we can check this by listing all of these six permutations of (B, C, D):

(B, C, D), (B, D, C), (C, B, D), (C, D, B), (D, B, C), (D, C, B).

Each of these six outcomes of Part (a) morph into the single combination/set outcome {B, C, D} for Part (b). In the same fashion, every other outcome of Part (b) will correspond to six different (permutations) of Part (a), and there is no overlapping since different outcomes of Part (b) will correspond to different sets. Therefore, if we (temporarily) let Y denote the answer to Part (b), and X denote the answer to Part (a), it follows that $6Y = X$, so that $Y = X/6 = 24/6 = 4$. Again, because of the size of this example, it is easy to get this answer directly (each of the four outcomes corresponds to deciding which of the four women not to hire). These ideas are easily generalized, and this is what we do next.

DEFINITION 5.1: A **permutation** of a set of distinct objects is any rearrangement of them (as an ordered list). More generally, if $1 \le k \le n$, a ***k*-permutation** of a set of n distinct objects is any permutation of any k of these n objects.

In the preceding example, we saw a way to count the 24 3-permuations of the set of four women {A, B, C, D}. The next result generalizes this idea.

THEOREM 5.4: The number of k-permutations taken from a set of n distinct objects $(1 \le k \le n)$ is denoted by $P(n,k)$ and is given by the following formula:

$$P(n,k) = n(n-1)(n-2)\cdots(n-k+1). \tag{2}$$

(Note that the number of factors is k.) This number is read as **the number of permutations of *n* objects taken *k* at a time**.

Proof: This is a simple application of the multiplication principle. We may view the task of forming a k-permutation as the k-step process of filling in the k ordered slots with objects taken from the n distinct objects (where each object can be used only once), working, say, from left to right. For the first slot, we have n objects to choose from. Once one has been selected, we move on to the second slot, where we may put any of the remaining $n - 1$ objects. For the third slot there will be $n - 2$ choices, and so on, until we get to the kth slot, when there will remain $n - k + 1$ choices of objects (convince yourself of this!). □

The special case when $k = n$ is of such importance that it motivates the following definition.

DEFINITION 5.2: Let n be a positive integer. The number of permutations of n distinct objects, $P(n,n)$, by (2) equals $n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$. This number is denoted $n!$, and is called **the factorial of *n***, or just ***n* factorial**. By convention, we define $0! = 1$.

An easy manipulation of (2) allows us to rewrite $P(n,k)$ entirely in terms of factorials:

$$P(n,k) = n(n-1)(n-2)\cdots(n-k+1)$$
$$= \frac{n(n-1)(n-2)\cdots(n-k+1)[(n-k)(n-k-1)\cdots 2\cdot 1]}{[(n-k)(n-k-1)\cdots 2\cdot 1]} = \frac{n!}{(n-k)!}.$$

In summary:

$$P(n,k) = \frac{n!}{(n-k)!}. \tag{3}$$

Formula (3) is more useful for theoretical and analytic manipulations than for actually computing numbers of permutations. For example, to compute $P(1000,3)$ by (2) is easy: $1000\cdot 999\cdot 998$, whereas formula (2) would involve 2000 multiplications (if we were to compute the factorials directly).[5] Before giving some more examples of permutations, we first formally introduce combinations. After this is done we give some examples that will mix both concepts so as to help the reader to develop a sense to better distinguish between permutations and combinations.

[5] Of course, most mathematical software computing platforms have built-in functions for computing factorials, but factorials get so large so quickly that, as a general rule, it is best to use formula (2) rather than (3) when computing numbers of permutations.

DEFINITION 5.3: If $0 \le k \le n$, a ***k*-combination** of a set of n distinct objects is any (unordered) subset that contains exactly k of these objects.

In the preceding example, we previewed a general method for counting combinations when we counted the six 3-combinations of the set {A, B, C, D} of four women. Here is the general result:

THEOREM 5.5: The number of k-combinations taken from a set of n distinct objects $(0 \le k \le n)$ is denoted by $C(n,k)$ and is given by the following formula:

$$C(n,k) = \frac{P(n,k)}{k!} = \frac{n!}{k!(n-k)!}. \tag{4}$$

This number is read as **the number of combinations of *n* objects taken *k* at a time**, or simply as ***n* choose *k***.

Proof: The only 0-combination is the empty set, so theorem is true for $k = 0$ (both sides equal 1). Next we assume that $k > 0$. Each k-combination $\{x_1, x_2, \cdots, x_k\}$ is just a subset of size k from the set of n objects under consideration, and gives rise to $k!$ permutations of its elements, by Theorem 2.3 (i.e., $P(k,k) = k!$). Since k-permutations arising from different k-combinations must also be different (because their elements come from different sets), and since all k-permutations (of the n objects under consideration) must be obtainable in this way, we may conclude that $C(n,k) \cdot k! = P(n,k)$. Dividing this equation by $k!$, and then using (3) produces (4). □

Computing and Counting with Permutations and Combinations

As with formula (3), since the factorials get large so quickly, directly computing the factorials in (4) is not an efficient way to compute $C(n,k)$. In particular, this can cause problems in floating point arithmetic computing systems (see Chapter 5 of [Sta-04]). Computationally, it is best to cancel the largest factorial in the denominator of (4) with the same factors in the numerator. For example, we would compute $C(200,2)$ as follows:

$$C(200,2) = \frac{200!}{2! \cdot 198!} = \frac{200 \cdot 199 \cdot \cancel{198!}}{2! \cdot \cancel{198!}} = \frac{200 \cdot 199}{1 \cdot 2} = 19{,}900.$$

Most computing platforms have built-in functions for computing factorials, permutations, and combinations.

As promised, we now give several examples to help provide the reader with some intuition on distinguishing between permutations and combinations. The key question to ask when trying to decide whether permutations or combinations are relevant is whether order matters—if it does, we are dealing with permutations, while if order does not matter, combinations are relevant. As in the last section, two (or more) counting principles might need to be combined to solve a given problem.

EXAMPLE 5.11: Answer each of the following counting questions.
(a) A recent Honolulu Marathon had 24,265 participants. How many top three finishes are (theoretically) possible?
(b) In how many ways can a committee of two democrats and two republicans be formed from a group of 60 republican senators and 40 democratic senators?
(c) Answer the question of Part (b) with the additional requirement that democratic senators K and L refuse to serve together.
(d) In how many ways can four math books, two computer science books, and five economics books be arranged on a shelf?
(e) In how many ways can the books in Part (d) be arranged on a shelf if books of the same subject need to be grouped together?

SOLUTION: Part (a): Order clearly matters here: P(24265, 3) = $24265 \cdot 24264 \cdot 24263 \approx 1.485 \times 10^{13}$. (Of course, only a handful of these outcomes would be reasonably likely.)

Part (b): The order that the people are put on this committee is not important, so combinations are relevant here. We can form such a committee by first taking a 2-combination of the set of 60 republicans (this can be done in $C(60,2)$ ways), and then taking a 2-combination from the set of 40 democrats (this can be done in $C(40,2)$ ways). By the multiplication principle, there will be $C(60,2) \cdot C(40,2)$ = $(60 \cdot 59/2!)(40 \cdot 39/2!) = 1,380,600$ such committees.

Part (c): We need to modify our solution of Part (b) to answer the present question. The part that needs modification is in computing the number of ways of choosing two democrats. There are two natural (and disjoint) cases: either a committee with neither K nor L, or a committee with (exactly) one of K or L. There will be $C(38,2) = 38 \cdot 37/2 = 703$ committees of the first type (since with neither K nor L, 38 democratic senators remain), and $C(2,1) \cdot C(38,1)$ = $2 \cdot 38 = 76$ committees of the latter type. Thus, we will have a grand total of $(60 \cdot 59/2!)(703 + 76) = 1,378,830$ such committees.

Part (d): We have to arrange 4 + 2 + 5 = 11 different books in a row. Order clearly matters here: there are 11! = 39,916,800 permutations of these 11 books.

Part (e): We first treat each type of book as a "block." Thus we have three blocks: the M block consisting of the four math books, the C block consisting of the two computer science books, and the E block consisting of the five economics books. We can view arranging the books on the shelf as first arranging the three blocks, for which there are 3! ways to do this, and then deciding how to permute the books in each block: for the M block, there are 4! ways to arrange the four math books, and similarly, there are 2! and 5! ways to arrange the books in the C and E blocks, respectively. Therefore, by the multiplication principle, the total number of ways of arranging the books in this fashion will be $3!\cdot 4!\cdot 2!\cdot 5! = 34{,}560$.

EXERCISE FOR THE READER 5.7: (*Circular permutations*) In Chinese restaurants, tables are often circular (so that everyone has an equally prominent seat), with a Lazy Susan in the center to facilitate access of the meal items; see Figure 5.7.

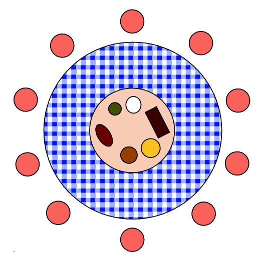

FIGURE 5.7: A Chinese dinner table for Exercise for the Reader 5.7.

(a) If there are n seats around the table, in how many ways could n people be seated around the table? Two seating arrangements are considered to be equivalent if the relative positions of all people are the same, i.e., if one arrangement can be obtained from the other by a rotation.
(b) How many arrangements are possible if $n = 2k$, where there are k men and k women and each man has a woman on either side?
(c) How many arrangements are possible as in Part (b) under the additional requirement that Jimmy and Sue should be seated next to one another?
(d) How many arrangements are possible as in Part (b) under the additional requirement that Jimmy and Sue should **not** be seated next to one another?

Our next example concerns five-card poker hands, which are assumed to be randomly drawn cards from a (shuffled) standard deck of playing cards.[6]

EXAMPLE 5.12: A poker hand consists of a random drawing of five cards from the standard 52-card deck.
(a) How many different poker hands are possible?
(b) How many poker hands are possible that contain at least one pair?

(d) How many poker hands are possible that contain no aces?
(e) How many poker hands are possible that contain at least one ace?

SOLUTION: Since order does not matter in poker hands, we are dealing with combinations in each part.

[6] The cards of a standard 52-card deck are evenly divided among the four *suits* (*clubs* and *spades*, which are black, and *diamonds* and *hearts*, which are red), and each suit has 13 denominations: (A)ce = 1, 2, …, 9, 10, (J)ack, (Q)ueen, (K)ing. The last three cards are *face cards*. A (five-card) *poker hand* is usually described in the most complimentary terms among the following possibilities, listed in order from least valuable (most common) to most valuable (most rare): high card, pair, two (separate) pairs, three of a kind, straight (five cards in sequence, ace can go before 2 or after king), flush (five cards of the same suit), full house (three of a kind and a pair), four of a kind, straight flush (straight plus flush), royal flush (10, J, Q, K, A, all of same suit). Thus, an example of a pair would be {8, 8, 2, 4, K}. Order does not matter in a poker hand.

Part (a): $C(52,5) = 52\cdot 51\cdot 50\cdot 49\cdot 48/(5\cdot 4\cdot 3\cdot 2\cdot 1) = 2{,}598{,}960$.

Part (b): We use the complement principle (often useful for counting sets that are described using the phrase "at least"). The number of poker hands that contain at least one pair is the total number of poker hands less the number that contain no pairs. The number with no pairs can easily be counted using the multiplication principle: There are $C(13,5)$ ways to choose five different denominations in such a poker hand and for each denomination we have four choices for the suit. It now follows that the number of poker hands that contain at least one pair is given by:

$$\mathrm{C}(52,5) - \mathrm{C}(13,5)\cdot 4^5 = 1{,}281{,}072.$$

Part (c): First, we choose the denomination for the pair; there are 13 choices. Next, from the four cards of this given denomination, we need to choose two, and there are $C(4,2) = 4\cdot 3/(1\cdot 2) = 6$ ways to do this. Once this is done we need to choose three different denominations for the remaining cards. There are $C(12,3)$ $12\cdot 11\cdot 10/1\cdot 2\cdot 3 = 220$ ways to do this. Finally for each of these chosen denominations, we have to choose one of the 4 cards.

$$\mathrm{C}(13,1)\cdot \mathrm{C}(4,2)\cdot \mathrm{C}(12,3)\cdot 4^3 = 13\cdot 6\cdot 220\cdot 4^3 = 1{,}098{,}240.$$

Part (d): Thinking of the poker hand as being dealt from a deck with the four aces removed, this gives the total number of such hands to be:

$$C(48,5) = 48\cdot 47\cdot 46\cdot 45\cdot 46/(5\cdot 4\cdot 3\cdot 2\cdot 1) = 1{,}712{,}304.$$

Part (e): We give two methods for counting these poker hands:

Method 1: (*Direct counting*) We decompose into disjoint cases:
Number of hands with at least one ace = Number with exactly one ace + Number with exactly two aces + Number with exactly three aces + Number with all four aces.
To count the number of hands with exactly two aces, we use the multiplication principle as follows:

$$\underbrace{C(4,2)}_{\substack{\text{Number of ways}\\ \text{to choose 2 aces}\\ \text{from the the four}}} \cdot \underbrace{C(48,3)}_{\substack{\text{Number of ways}\\ \text{to choose 3 cards}\\ \text{from the non-aces}}}$$

The same idea works for other cases to yield the grand total to be:

$$\underbrace{C(4,1)\cdot C(48,4)}_{\text{Hands with exactly one ace}} + \underbrace{C(4,2)\cdot C(48,3)}_{\text{Hands with exactly two aces}} + \underbrace{C(4,3)\cdot C(48,2)}_{\text{Hands with exactly three aces}} + \underbrace{C(4,4)\cdot C(48,1)}_{\text{Hands with all four aces}}$$

$$= 778{,}320 + 103{,}776 + 4512 + 48 = 886{,}656.$$

Method 2: (*Using complements*) The complement of the set of poker hands containing at least one ace is simply the set of poker hands with no aces. So we can get the answer we want by subtracting the number of hands with no aces (which we figured out in Part (d)) from the total number of poker hands (that we computed in Part (a)):

$$\underbrace{2{,}598{,}960}_{\substack{\text{Total number of}\\ \text{possible poker hands}}} - \underbrace{1{,}712{,}304}_{\substack{\text{Number of possible}\\ \text{poker hands without}\\ \text{aces}}} = \underbrace{886{,}656}_{\substack{\text{Number of possible}\\ \text{poker hands with at}\\ \text{least one ace}}}$$

This provides a nice check. In general, complements can often save time for counting problems involving phrases such as "at least" or "no more than."

EXERCISE FOR THE READER 5.8: In a standard five-card poker hand, compute the total number of possible five-card poker hands that are:
(a) full houses.
(b) flushes.
(c) four of a kind hands.

At this point, it will be beneficial to observe a useful identity for combination numbers $C(n, k)$. Since $C(n, k)$ represents the total number of k-element subsets of a set with n elements, it follows that $\sum_{k=0}^{n} C(n,k)$ must be the total number of subsets of a set with n elements. But we know from Theorem 5.1 that this latter number is just 2^n. We have thus proved the following theorem:

THEOREM 5.6: The number of subsets of a set with n elements is

$$C(n,0)+C(n,1)+\cdots+C(n,n)=2^n. \tag{5}$$

The Binomial Theorem

This is one of many combinatorial identities involving k-combination coefficients. Our next result is a theorem of algebra involving these coefficients. In such algebraic contexts, the k-combination coefficients are customarily referred to as *binomial coefficients*, and are denoted as follows:

NOTATION: For nonnegative integers n and k, with $k \le n$, the **binomial coefficient** $\binom{n}{k}$ is the number $C(n,k)$ of n objects taken k at a time, i.e., (from (4)):

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

THEOREM 5.7: (*The Binomial Theorem*) If x and y are any numbers, and n is a nonnegative integer, then

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}. \tag{6}$$

Proof: There are several ways to prove this theorem; keeping in the spirit of this chapter, we will present a combinatorial proof. If we expand the left side of (5)

$$(x+y)^n = \underbrace{(x+y)(x+y)\cdots(x+y)}_{n \text{ factors}},$$

we can view the result as a sum of terms, each one arising by choosing either an x or a y from each of the n factors. Thus, each term that will arise must be of the form $x^k y^{n-k}$ for some nonnegative integer k between 0 and n (inclusive), and this will correspond to choosing x's from k of the factors, and y's from the remaining $n-k$ factors. Now, in how many such ways can the term $x^k y^{n-k}$ arise? This will simply be the number of ways that one can choose a set of k of the n factors to be designated as x-factors (and the remaining $n-k$ factors to be designated as y-factors). Since the order in which these factors are selected is unimportant (the product of the x's and y's will always work out to be $x^k y^{n-k}$), this number is simply $C(n,k) = \binom{n}{k}$. This completes the proof of (6). □

EXAMPLE 5.13: Use the binomial theorem to expand the following square and cubic polynomials: (a) $(x+y)^2$, and (b) $(a-2b)^3$.
(c) What is the coefficient of x^{10} in the expansion of $(x+2)^{16}$?

SOLUTION: Part (a): Using (6) directly with $n = 2$, we obtain

$$(x+y)^2 = \binom{2}{0} x^0 y^{2-0} + \binom{2}{1} x^1 y^{2-1} + \binom{2}{2} x^2 y^{2-2} = y^2 + 2xy + x^2 .$$

Part (b): Using (6) with $n = 3$ (and with $x = a$ and $y = -2b$), we obtain

$$\begin{aligned}(a-2b)^3 &= \binom{3}{0} a^0 (-2b)^{3-0} + \binom{3}{1} a^1 (-2b)^{3-1} + \binom{3}{2} a^2 (-2b)^{3-2} + \binom{3}{3} a^3 (-2b)^{3-3} \\ &= -8b^3 + 3a \cdot 4b^2 + 3a^2(-2b) + a^3 \\ &= a^3 - 6a^2 b + 12ab^2 - 8b^3\end{aligned}$$

Part (c): The term in the right side of (6) corresponding to x^{10} would correspond to the index $k = 10$. The resulting coefficient (with $n = 16$ and $y = 2$ and removing x^{10}) is thus

$$\binom{16}{10} \cdot 2^{16-10} = \frac{16!}{10!6!} \cdot 2^6 = \frac{11 \cdot \cancel{12} \cdot 13 \cdot 14 \cdot \cancel{15} \cdot 16}{1 \cdot \cancel{2} \cdot \cancel{3} \cdot \cancel{4} \cdot \cancel{5} \cdot \cancel{6}} \cdot 2^{\cancel{6}4} = 512{,}512.$$

One very attractive way to obtain binomial coefficients is from the following so-called **Pascal's triangle** that is often first introduced in high school algebra courses:

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1 \\
\iddots & & & & \vdots & & & & \ddots
\end{array}
\qquad
\begin{array}{ccccccccc}
 & & & & \binom{0}{0} & & & & \\
 & & & \binom{1}{0} & & \binom{1}{1} & & & \\
 & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
 & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\
\binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} \\
\iddots & & & & \vdots & & & & \ddots
\end{array}
$$

The triangle of numbers goes on forever. Each of the outer diagonal entries is 1, corresponding to the identity $\binom{n}{0}=\binom{n}{n}=1$. Each internal entry is obtained by adding the two entries that lie immediately to the above left and right of it. This follows from the following identity for binomial coefficients:

$$\binom{n-1}{k-1}+\binom{n-1}{k}=\binom{n}{k}, \tag{7}$$

valid whenever $0<k<n$.

Proof of (7): We can give a nice combinatorial proof of (7) as follows: The right side of (7) is the number of different subsets of size k that one can choose from a set of n objects. We consider a set T with n elements and remove one of the elements, which we label as a, and we call the resulting set S (for smaller set). Thus, we can write $T=S\cup\{a\}$, where the union is disjoint (so S has $n-1$ elements). We know that $\binom{n}{k}$ corresponds to the number of subsets of T having k elements. Now, such a subset can either be a subset of S also with k elements (if it does not contain a), or consists of a together with a subset of S of size $k-1$ (if it does contain a). There are $\binom{n-1}{k}$ subsets of the first type (corresponding to subsets of S having k elements) and $\binom{n-1}{k-1}$ subsets of the second type (corresponding to subsets of S having $k-1$ elements). Since these two types of sets form a disjoint partition of the size k subsets of T, the identity (7) follows. □

EXERCISE FOR THE READER 5.9: Give a non-combinatorial proof of the identity (7) using factorial manipulations.

EXERCISE FOR THE READER 5.10: (a) Prove the following identity:

$$1 = \binom{m}{1} - \binom{m}{2} + \binom{m}{3} - \cdots \pm \binom{m}{m},$$

where m is any positive integer.
(b) Use the identity of Part (a) to prove the general case of the inclusion-exclusion principle (formula (1) from the last section):

$$\begin{aligned} |A_1 \cup A_2 \cup \cdots \cup A_n| &= \sum_{i=1}^{n} |A_i| - \sum_{i_1 < i_2} |A_{i_1} \cap A_{i_2}| + \cdots \\ &\quad + (-1)^{a+1} \sum_{i_1 < i_2 < \cdots < i_a} |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_a}| + \cdots \\ &\quad + (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n|. \end{aligned}$$

Suggestion: For Part (a), use the binomial theorem. For Part (b), consider a single element $x \in \cup A_i$, let m be the number of sets A_i to which x belongs. Use the identity of Part (a) to count the number of times the right side of (1) contributes to the count of the element x.

Multinomial Coefficients

The binomial coefficients are a special case of the so-called *multinomial coefficients*, which we motivate with the following example:

EXAMPLE 5.14: (*Motivating example for multinomial coefficients*) How many "words" can be created by rearranging the four letters in the word "look?" Here we take a word in the general sense to mean any sequence of four letters, regardless whether it has any meaning in any language.

SOLUTION: If the letters in "look" were all different, the answer would simply be the number of permutations of four objects, or $4! = 24$. Let us temporarily label the duplicate letter o's as o_1 and o_2, so they will be distinct. Then, any of the 24 possible permutations of the list "l o_1 o_2 k," say "k o_1 o_2 l," can have the two symbols o_1 and o_2 permuted (k o_2 o_1 l) and the result will be indistinguishable from the original permutation, once the artificial labels on the o's have been removed. On the other hand, any other permutation of "k o_1 o_2 l" would be distinguishable if the o-labels are detached. Thus, to get the number of distinguishable permutations of "look," we need to divide the total number of permutations 4! by this duplication number 2!, to get 4!/2! = 12.

We now state and prove the general result:

THEOREM 5.8: (*Permutations of Objects That Are Not All Distinguishable*) Suppose that n objects are of k different types. Assume that there are n_1 objects of

Type 1, n_2 are of Type 2, …, and n_k objects of Type *k*, where $n = n_1 + n_2 + \cdots + n_k$. The number of distinguishable permutations of these *n* objects is given by the **multinomial coefficient**

$$\binom{n}{n_1,\, n_2,\, \cdots,\, n_k} \equiv \frac{n!}{n_1!\, n_2! \cdots n_k!}.$$

This number also coincides with the number of ways to place *n* distinct objects into *k* distinguished groups with n_1 objects in the first group, n_2 in the second group, …, and n_k in the last group.

We will give two different combinatorial proofs of this theorem; the first proceeds along the lines of the motivating example, while the second is a more direct proof.

Proof 1: We know that there are $n! = (n_1 + n_2 + \cdots + n_k)!$ permutations of these objects. For any such permutation, any of the n_1 objects of Type 1 can be permuted in any of the $n_1!$ possible ways and the resulting permutation will not be distinguishable from the original. The same holds true if we perform any of the $n_2!$ possible permutations of the Type 2 objects, any of the $n_3!$ possible permutations of the Type 3 objects, and so on. By the multiplication principle, it follows that each permutation thus corresponds to a total of $n_1! \cdot n_2! \cdots n_k!$ permutations that are indistinguishable from one another. These sets of permutations partition the entire collection of permutations. Dividing the total number of permutations by this duplication number gives us the asserted number of distinguishable permutations. The latter statement can be justified in the same fashion. □

Proof 2: Imagine a row of *n* slots to fill with these *n* objects.
Choose n_1 slots to be filled with objects of Type 1. There are $\binom{n}{n_1}$ ways to do this. From the remaining $n - n_1$ slots, we next choose n_2 of them to be filled with objects of Type 2. This can be done in $\binom{n-n_1}{n_2}$ ways. From the remaining $n - n_1 - n_2$ slots, we next choose n_3 of them to be filled with objects of Type 3. This can be done in $\binom{n-n_1-n_2}{n_3}$ ways. Continuing in this fashion, the multiplication principle tells us that the number of distinguishable permutations of these *n* objects is given by:

$$\binom{n}{n_1}\cdot\binom{n-n_1}{n_2}\cdot\binom{n-n_1-n_2}{n_3}\cdots\cdot\binom{n-n_1-n_2-\cdots-n_{k-1}}{n_k}$$

$$=\frac{n!}{n_1!\,\cancel{(n-n_1)!}}\cdot\frac{\cancel{(n-n_1)!}}{n_2!\,\cancel{(n-n_1-n_2)!}}\cdot\frac{\cancel{(n-n_1-n_2)!}}{n_3!\,\cancel{(n-n_1-n_2-n_3)!}}$$

$$\cdots\cdots\frac{\cancel{(n-n_1-n_2-\cdots-n_{k-1})!}}{n_k!(n-n_1-n_2-\cdots-n_{k-1}-n_k)!}$$

$$=\frac{n!}{n_1!n_2!n_3!\cdots n_k!0!}=\frac{n!}{n_1!\,n_2!\cdots n_k!}.$$

The latter statement can be justified in the same fashion. □

EXERCISE FOR THE READER 5.11: How many distinguishable permutations are there of the word MISSISSIPPI?

In the basic Example 5.9, we would have n = 4 (four letters), $n_1=n_3=1$ (corresponding to the unique letters "l" and "k") and $n_2=2$ (corresponding to the duplicated letter "o"), and the answer we obtained equals $\binom{4}{1,\,2,\,1}$ $=\frac{4!}{1!\,2!\,1!}=\frac{24}{1\cdot 2\cdot 1}=12$. The multinomial coefficients are so named because of the multinomial theorem that we will describe shortly. Note that when k = 2, the multinomial coefficient $\binom{n}{n_1,\,n_2}$ coincides with the binomial coefficients $\binom{n}{n_1}=\binom{n}{n_2}$.

The combinatorial proof that we gave for the binomial theorem generalizes naturally to the expansions of powers of multinomials. The general result is contained in the following theorem:

The Multinomial Theorem

THEOREM 5.9: (*The multinomial theorem*) If $x_1,x_2,\cdots,x_r$ are numbers, and n is a nonnegative integer, then

$$(x_1+x_2+\cdots+x_r)^n=\sum_{\substack{k_1+k_2+\cdots+k_r=n\\ k_i\ \text{nonnegative integer}}}\binom{n}{k_1,\,k_2,\cdots,k_r}x_1^{k_1}x_2^{k_2}\cdots x_r^{k_r} \qquad (8)$$

The sum in the right-hand side of (8) can be viewed as being taken over all of the vectors $(k_1, k_2, \cdots, k_r)$ of nonnegative integers ($0 \le k_i \le n$) whose components add up to n.

EXERCISE FOR THE READER 5.12: Prove Theorem 5.9.

EXAMPLE 5.15: Use the multinomial theorem to expand $(x+2y+3z)^2$.

SOLUTION: We have r = 3 (trinomial) and n = 2 (with $x_1 = x,\ x_2 = 2y$ and $x_3 = 3z$). The totality of vectors (k_1, k_2, k_3) corresponding to the terms in the right-hand side of (8) are as follows: (k_1, k_2, k_3) = (2,0,0), (0,2,0), (0,0,2), (1,1,0), (1,0,1), and (0,1,1). The first three of the corresponding multinomial coefficients all equal $2!/2! = 1$, while the last three equal $2!/1!1! = 2$. Computing with (8) (in the order that these vectors were listed), we now obtain

$$\begin{aligned}(x+2y+3z)^2 &= x^2 + (2y)^2 + (3z)^2 + 2[x(2y) + x(3z) + (2y)(3z)]\\ &= x^2 + 4y^2 + 9z^2 + 4xy + 6xz + 12yz.\end{aligned}$$

EXERCISE FOR THE READER 5.13: Show how formula (8) can specialize to the binomial theorem (5).

EXERCISE FOR THE READER 5.14: What is the coefficient of $a^6b^3c^3d^2$ in the expansion of $(2a-3b+4c-d)^{14}$?

We end this section with another useful counting argument. At first glance, combinations might not seem very relevant, but with an ingenious artifice they may indeed be applied.

EXAMPLE 5.16: (*Motivating example for a partitioning argument*) Joey has five identical chocolate bars that he plans to give to his three cousins, Abby, Billy, and Christy. In how many different ways can he distribute these bars to his cousins?

SOLUTION: The different distributions of the five chocolate bars can be displayed graphically by laying out the five chocolate bars in a horizontal row, and inserting two partitions anywhere among the six slots between bars (or to the left/right of all of them). This will partition the bars into three groups (some possibly empty): the group to the left of the two partitions, which we arbitrarily assign to be Abby's allotment, the group between the two partitions: Billy's allotment, and the group to the right of the two partitions: Christy's allotment. Figure 5.8 shows a particular allotment with this scheme. Note that there is one less partition bar than the number of people to distribute to.

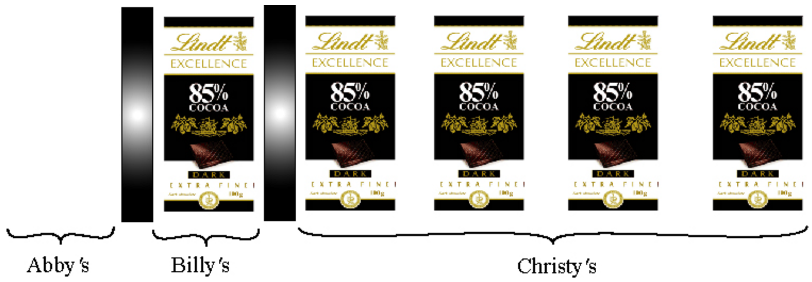

FIGURE 5.8: A possible distribution for the five (identical) chocolate bars to three people. The two barriers partition the distribution into three categories: (i) to the left of the first partition: no bars to Abby, (ii) between the two barriers: one bar to Billy, and (iii) to the right of the second partition: four bars to Christy.

Clearly, the different allotments of the five bars to the three individuals correspond to the number of different bar/partition diagrams. We can view this question as the count of the number of distinguishable arrangements of seven objects (five chocolate bars + two partition bars), where the five chocolate bars are identical as are the two partition bars. Thus, by Theorem 5.8, the number of such arrangements is $\binom{5+(3-1)}{5,\ 2} = \binom{7}{2} = 21$.

The reader should be able to prove the following general result, the task will be left as Exercise 36.

THEOREM 5.10: (*Distribution of Identical Objects to Different Places*) The number of ways to distribute n identical objects to d different (distinguishable) places is given by $\binom{n+(d-1)}{d-1}$.

EXERCISE FOR THE READER 5.15: (a) How many different nonnegative integer solutions are there (for x_1, x_2, x_3, x_4) in the equation

$$x_1 + x_2 + x_3 + x_4 = 12?$$

(b) How many different positive integer solutions are there for the equation in Part (a)?

Suggestion: For Part (a), consider the analogy of placing 12 identical balls into 4 different urns and view assigning each x_i a nonnegative integer as placing this number of balls into the *i*th urn. For Part (b), introduce new variables $y_i = x_i + 1$ which will be positive integers whenever the x_i's are nonnegative integers, and then use the method of Part (a).

EXERCISE FOR THE READER 5.16: How many terms are there in the sum (8) of the multinomial theorem?

EXERCISES 5.2:

1. (a) Write down all of the permutations of the word CAT.
(b) Write down all 2-permutations of the objects $\{A, B, C, D, E\}$.
(c) Write down all 2-combinations of the objects $\{A, B, C, D, E\}$.

2. (a) Write down all of the permutations of the list (1, 2, 3).
(b) Write down all 3-permuations of the objects {1, 2, 3, 4, 5}.
(c) Write down all 3-combinations of the objects {1, 2, 3, 4, 5}.

3. Compute each of the following quantities:

(a) $P(3,3)$	(b) $C(5,5)$	(c) $P(52,3)$
(d) $C(20,5)$	(e) $P(5,1)$	(f) $C(6,0)$

4. Compute each of the following quantities:

(a) $P(6,3)$	(b) $C(5,2)$	(c) $P(100,5)$
(d) $C(200,5)$	(e) $P(8, 8)$	(f) $C(1000,0)$

5. *Pizza Castle* offers 12 different toppings on their pizzas.
(a) How many different three-topping pizzas can be ordered?
(b) How many different pizzas can be made that include up to three toppings (no toppings is possible, this would be just a plain cheese pizza)?

6. *Cold Cream* offers its ice cream sundae with a choice of one, two, or three scoops of ice cream, and a choice of exactly three different toppings from eight available toppings. (Regardless of the number of scoops ordered, three different toppings must be chosen.)
(a) How many different sundaes can be ordered with vanilla ice cream?
(b) How many different sundaes can be ordered if the ice cream can be chosen from 31 different flavors, but it must be the same flavor for each scoop?
(c) How many different sundaes can be ordered if the ice cream can be chosen from 31 different flavors, and can be a different flavor for each scoop?

7. (a) Suppose you have won five tickets to an upcoming LA Lakers basketball game. In how many ways can you invite 4 of your 12 best friends to come along?
(b) Suppose you have seven different NBA basketball team T-shirts. In how many ways can you distribute these shirts to 7 of your 12 best friends?

8. The computer science (CS) faculty at a certain university consists of 20 members, 13 of whom are senior faculty and 7 of whom are junior members. The total number of faculty at this university is 365. A hiring committee for a new computer science faculty member is to be composed of five faculty members. How many hiring committees are possible if:
(a) They all come from the CS department?
(b) The committee is composed of three senior and two junior CS faculty members?
(c) The committee is composed of CS faculty with at least one junior member?
(d) The committee contains exactly one outside (the computer science department) faculty member, at least one senior CS faculty member, and at least one junior CS faculty member?

9. A math department is giving out awards to a particularly strong senior class. It has three different awards: Award I, worth \$2000, Award II, worth \$1000, and Award III, worth \$500. It decides that it can distribute a total of five awards, to five students from the group of eight outstanding students A, B, C, D, E, F, G, and H. In how many ways can the awards be distributed if:
(a) Five Award III's will be distributed.
(b) Any combination of the three awards can be given.
(c) Any combination of the three awards can be given, but students B and H should either both

get an award or both not get an award.
(d) Any combination of the three awards can be given, but students B and H should either both get the same award or both not get an award.

10. The French club at a certain university has 12 active members, 5 of whom are men. A yearbook photo will be taken with 8 of the 12 students lined up in a row. How many such photo arrangements are possible if:
(a) There are no restrictions.
(b) Half of the subjects must be women.
(c) Half of the subjects must be women, the men all stand together, and the women all stand together.
(d) Half of the subjects must be women, and the men should all stand together.
(e) Half of the subjects must be women, and no two men stand together.
(f) If there are more women than men, no two women stand together; otherwise no two men stand together.

11. At a five-year reunion of a college tennis team, 26 former teammates show up, 15 of whom are men.
(a) If everyone shakes everyone else's hand, how many handshakes will there be?
(b) If all men shake hands with one another and all the women hug one another, how many hugs and handshakes will there be?
(c) If all women shake hands with one another and hug all of the men, how many hugs and handshakes will the women be involved with?

12. Count the number of possible three-card poker hands (consisting of three cards drawn from a shuffled standard deck of 52 cards) that contain:
(a) At least two spades.
(b) A flush (three cards of the same suit).
(c) Three different suits.

13. Count the number of possible four-card poker hands (consisting of four cards drawn from a shuffled standard deck of 52 cards) that contain:
(a) At most one heart.
(b) Four different suits.
(c) At least two cards of the same suit.

14. In the senatorial primary election of a certain year in Guam, there are 23 Democratic candidates and 16 Republican candidates. The rules for a voting ballot are that up to 15 candidates can be voted for, but only from one party. In how many ways can a ballot be (correctly) cast, assuming that voting for no candidates is an acceptable ballot (indeed, a political statement)?

15. To pass an exam, a law student must choose five of eight essay questions to answer. How many choices does he/she have? What if he/she is required to answer at least three of the first four questions?

16. How many permutations of the letters $\{A, B, C, D, E, F, G\}$ are there, such that:
(a) A precedes B?
(b) A precedes B, and C precedes D?
(c) A precedes B, which in turn precedes C?
(d) C, D, and E appear together in this order?
(e) A and B are appear together in this order, as do C and D?

17. How many permutations of the letters $\{A, B, C, D, E, F, G\}$ are there, such that:
(a) C and D are next to each other?
(b) A, B, and C are next to each other?
(c) C is between A and B?
(d) F, A, and D are seated together in this order?
(e) F, A, and D are seated together in this order, as are G and B.

18. (a) How many possible finishes are there in a three-car drag race if double and triple ties are possible?
(b) How many possible finishes are there in a four-car drag race if double, triple, and quadruple ties are possible?

19. (a) In how many different ways can 10 men be paired off to dance with 10 of 15 women?
(b) Same question as (a) but with the additional requirement that Jack either dances with Cindy or sits out (in which case only nine couples would dance).
(c) Same question as (a) but with the additional requirement that Jenny and Clair will either both dance or both sit out?

20. (a) Expand $(x-z)^5$.
(b) Expand $(2x+3y)^6$.
(c) What is the coefficient of x^{12} in the expansion of $(2x^3-5)^5$?

21. (a) Expand $(x+z)^7$.
(b) Expand $(5x+y^3)^5$.
(c) What is the coefficient of x^6y^6 in the expansion of $(3x^2-4y^3)^5$?

22. (a) Prove that for any positive integer *n*, we have $3^n = \sum_{k=0}^{n}\binom{n}{k}2^k$.
(b) Obtain a similar expansion for x^n for any real number *x*.
(c) From Part (b) obtain the expansion $\sum_{k=0}^{n}\binom{n}{k}(-1)^k = 0$.

23. Prove that for any positive integer *n*, we have $\sum_{k=0}^{n}\binom{n}{k}^2 = \binom{2n}{n}$.
Suggestion: An elegant combinatorial proof can be achieved by considering a set *T* containing 2*n* elements, and splitting *T* into two *n*-element sets *R* and *S*. Now any *k*-combination of *T* must be expressible as a disjoint union of a *j*-combination of *R* and an *n* – *j* combination of *S*, for some nonnegative integer *j*.

24. (a) Expand $(x+2y+3z)^4$.
(b) Expand $(x-y^2-z^3+w)^3$.
(c) What is the coefficient of $x^5y^3z^8$ in the expansion of $(5x-2y+3z^2)^{12}$?

25. (a) Expand $(2x-2y+5z)^4$.
(b) Expand $(x+y+z+w)^4$.
(c) What is the coefficient of $x^4y^6z^8w^{24}$ in the expansion of $(x+2y+3z^2+w^4)^{20}$?

26. How many distinguishable permutations are there of each of the following words?
(a) CANADA (b) SWEET
(c) BANANA (d) MATHEMATICS

27. How many distinguishable permutations are there of each of the following words?
(a) YOYO (b) LOLLIPOP
(c) ELEMENTAL (d) KAMEHAMEHA

28. Suppose that a boat runs colored flags up a vertical pole to make signals. The boat has three red flags, and six white flags.
(a) How many different signals can the boat's captain give using all nine of these flags?
(b) How many different signals can be made using exactly three flags? Assume that only the relative position of the flags matters, i.e., different gaps and/or positionings of the flags do not count as different signals.
(c) How many different signals can be made using from one to three flags? See Part (b) for the conventions.

29. Suppose that a boat runs colored flags up a vertical pole to make signals. The boat has three red flags, two green flags, and four yellow flags.
(a) How many different signals can the boat's captain give using all nine of these flags?
(b) How many different signals can be made using exactly three flags? Assume that only the relative position of the flags matters, i.e., different gaps and/or positionings of the flags do not count as different signals.
(c) How many different signals can be made using from one to three flags? See Part (b) for the conventions.

30. (a) In how many ways can 24 new (and identical) computers be distributed to the Math Laboratory, the Computer Laboratory, and the Physics Laboratory at a certain university?
(b) Same question as (a) but with the additional requirement that the Physics Laboratory must receive at least three computers, and the other two labs must receive at least one each.

31. Suppose that we have \$15,000 to invest in (up to) three different mutual funds: A, B, and C, and that we can allocate investments in each fund in increments of \$500.
(a) How many such investment allocations are possible?
(b) How many such allocations are possible if the minimum investments in funds A and C are \$2500?

32. (a) How many solutions are there of the equation $x_1 + x_2 + x_3 + x_4 = 10$, where each $x_i \geq -1$ an integer?
(b) Repeat Part (a) if now only $x_1, x_3 \geq -1$ but $x_2, x_4 \geq -2$.

33. *Pumpkin's Donuts* sells eight different kinds of donuts. How many different dozens of donuts can be sold?

34. A wallet contains five traveler's checks that are taken from the following denominations: \$1, \$5, \$50, \$500.
(a) How many different combinations of travelers checks are possible?
(b) Do different combinations always result in different total dollar amounts?

35. (a) Give a combinatorial proof of the identity $\binom{n}{n-k} = \binom{n}{k}$, $0 \leq k \leq n$, and then give a noncombinatorial proof.
(b) For a positive integer n, what is the value of the integer k, $0 \leq k \leq n$, for which $\binom{n}{k}$ is at its maximum value? Show that the binomial coefficients increase as k increases to this value, and then decrease as k increases from this value to n.
Note: The identity in Part (a) corresponds to the left-right symmetry in Pascal's triangle.
Suggestion: For Part (b), deal separately with the cases in which n is even and n is odd.

36. Use mathematical induction to give another proof of the binomial theorem.

37. How many onto functions *f* are there with the following domains and codomains?
(a) $f: \{1,2,3,4,5\} \to \{1,2,3,4,5,6\}$ (b) $f: \{1,2,3,4,5,6\} \to \{1,2,3,4,5,6\}$

(c) $f:\{1,2,3,4,5,6\}\to\{1,2,3,4,5\}$ (d) $f:\{1,2,3,4,5,6,7\}\to\{1,2\}$

38. (a) If A is a finite set with n elements, show there are 2^n-2 different onto functions $f:A\to\{1,2\}$.
(b) How many onto functions are there of the form $f:\{1,2,3,4,5,6,7\}\to\{1,2,3\}$?
Suggestion: For Part (a) notice that any nonconstant function is onto. For Part (b), use the result of Part (a) in counting separately the (disjoint) cases where $|f^{-1}(\{3\})|=1,2,3,4$, or 5.

39. Prove that for nonnegative integers n, m, and r, with $0\le r\le \min(n,m)$, we have
$$\sum_{k=0}^{r}\binom{n}{k}\binom{m}{r-k}=\binom{n+m}{r}.$$
Suggestion: This result generalizes that of Exercise 23; the suggestion given there can be modified for the present needs.

40. Prove Theorem 5.9.

41. Prove the following identity: $\sum_{k=0}^{n}k\binom{n}{k}=n\cdot 2^{n-1}$.
Suggestion: An elegant combinatorial proof can be achieved by counting the number of ways to form a committee (from a group of n individuals), along with a distinguished chairperson. The left and right sides of the identity outline two different schemes for counting the total number of such committees.

42. Prove the following identity: $\sum_{k=0}^{n}k\binom{n}{k}^2=n\cdot\binom{2n-1}{n-1}$.
Suggestion: A combinatorial proof can be given using an idea similar to that given in the suggestion of the preceding exercise. This time, the committee is formed from two separate groups of individuals and the chairperson is taken from the first group.

43. Give a combinatorial proof of the following identity, which is valid for $1\le k\le n$:
$$k\binom{n}{k}=n\binom{n-1}{k-1}.$$

44. Give a combinatorial proof of the following identity, which is valid for $1\le k\le n$:
$$\binom{n}{k}=\sum_{j=k}^{n}\binom{j-1}{k-1}.$$

5.3: GENERATING FUNCTIONS

In this section we will develop a very effective tool for analyzing sequences relating to combinatorial problems. This tool is based on storing the terms of the sequence as coefficients of a formal power series. The resulting formal power series is called the *generating function* for a given sequence. Generating functions can be manipulated by a set of natural rules, which are motivated by the ordinary arithmetic of polynomials, allowing one to operate on the whole sequence at once, and these concepts give them surprising power that can be used to solve seemingly intractable problems. Many useful properties about the convergence of power series are proved in calculus books. This section will develop generating functions from a non-calculus perspective. Although some facts will be "borrowed" from calculus, our treatment will be entirely self-contained. We will thus bypass details concerning the convergence of series, and merely perform formal manipulations on them and show how to apply such manipulations to solve an assortment of combinatorial problems.

Generating Functions and Power Series

DEFINITION 5.4: For any sequence $a_0, a_1, a_2, \cdots$ of real numbers, the corresponding **(ordinary) generating function** of the sequence is the following formal infinite power series:

$$G(x) = a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{n=0}^{\infty} a_n x^n.$$

We use the function notation on the left of the above equation, even though the infinite series may not define much of a function. Such an infinite series of increasing nonnegative powers of the variable x with real number coefficients is called a **power series**.

NOTE: Power series are studied in standard calculus courses (usually in the second semester). If we substitute $x = 0$ into <u>any</u> power series, the series becomes $a_0 + a_1 \cdot 0 + a_2 \cdot 0^2 + \cdots = a_0 + 0 + 0 + \cdots = a_0$. Although it is impossible to perform the infinite number of additions required in a general power series (when a nonzero number x is substituted), there are circumstances when the infinite sum makes sense (in which case we say it *converges*), and in such a case it can be evaluated to any degree of accuracy by adding up a sufficiently large finite number of its terms. For any power series there always exists a number R (called the *radius of convergence* of the power series) in the range $0 \le R \le \infty$, such that the power series converges for all x in the range $|x| < R$, but does not converge for any x in the range $|x| > R$. It is possible that $R = 0$ (in which case the series converges only when $x = 0$), but in all other cases the power series truly equals some function of x within the radius of convergence. In such cases, we identify the series with the function that it defines, i.e., both are considered to be the generating function of the sequence.

In this section the adjective "ordinary" (for generating functions) will be redundant since this will be the only sort of generating functions that will be considered. More extensive treatments consider other useful generating functions, such as exponential generating functions; see, for example, [Wil-90]. Throughout this section we shall treat generating functions essentially as formal objects, not concerning ourselves with the question of whether the infinite series converges for any nonzero numbers x.[7] We will present closed formulas for a few key generating functions (i.e., the function formula equals the power series within a positive radius of convergence), but the calculus details will be omitted.

EXAMPLE 5.16: Determine the generating functions of the following sequences:

(a) $a_n = 1\ (n = 0,1,2,\cdots)$

(b) $a_n = n!\ (n = 0,1,2,\cdots)$

(c) $a_0 = 1,\ a_1 = 1,\ a_2 = 1,\ a_3 = 1$

NOTE: For any *finite* sequence as in Part (c), in order to form the generating function, the remaining terms are assumed to be zero. Thus the power series will just be a finite sum, and will define a polynomial.

SOLUTION:

Part (a): The generating function is $G(x) = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n$. Example 5.18 presents an explicit formula for the function defined by this power series.

Part (b): The generating function is $G(x) = 1 + x + 2x^2 + \cdots + n!x^n + \cdots = \sum_{n=0}^{\infty} n!x^n$.

Part (c): The generating function is the polynomial $G(x) = 1 + x + x^2 + x^3$.

Algebraic identities sometimes allow us to express polynomials (finite power series) as single mathematical expressions. The next example will provide a few such simplifications.

EXAMPLE 5.17: For each part, express the polynomial generating function defined by the sequence of coefficients as a single-term closed-form algebraic expression:

(a) $a_0 = 1,\ a_1 = 2,\ a_2 = 4,\ a_3 = 8,\ a_4 = 16$

(b) $a_0 = C(n,0), a_2 = C(n,1), \cdots, a_n = C(n,n)$, where n is a positive integer.

SOLUTION:

Part (a): The generating function for the given sequence is $1 + 2x + 4x^2 + 8x^3 + 16x^4 = (2x)^0 + (2x)^1 + (2x)^2 + (2x)^3 + (2x)^4$, which is a finite geometric series (see Proposition 3.5 of Section 3.1), so by formula (3) of Chapter 3, it can be rewritten as: $((2x)^5 - 1)/(2x - 1) = (32x^5 - 1)/(2x - 1)$.

Part (b): The generating function for the given coefficients is $\sum_{k=0}^{n} C(n,k)\, x^k$. Using the binomial theorem (formula (6) of Theorem 5.7):

[7] For readers who have studied calculus, when the series converges for nonzero values of x, the resulting function will coincide with the generating function, and the series will be its so-called Taylor series. This gives another way to view the coefficients in terms of derivatives of the generating functions.

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = \sum_{k=0}^{n} C(n,k)\ x^k y^{n-k},$$

and substituting $y = 1$, gives us $(1+x)^n = \sum_{k=0}^{n} C(n,k)\ x^k$. The left side is the factored form of the polynomial generating function for the sequence of binomial coefficients.

Such algebraic conversions for polynomials in the above example can be extended to a great variety of infinite power series by making use of some key (building block) closed-form expressions for power series that will be borrowed from calculus. Our next example provides an important generating function for a very simple sequence: the sequence of constant 1s:

EXAMPLE 5.18: Part (a): In calculus courses, it is proved that the generating function of the infinite sequence of 1s: $a_0 = 1, a_1 = 1, a_3 = 1\cdots,$ is the function $1/(1-x)$, and furthermore that the power series equals the function when $|x| < 1$:

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n. \tag{8}$$

Another useful generating function corresponds to the sequence of reciprocal factorials: $a_n = 1/n!$, and is the exponential function e^x. The following equality is valid for all real numbers x:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}. \tag{9}$$

Part (b): Use (8) to find a closed expression for the generating function of the sequence $a_n = \begin{cases} 1, \text{ if } n \text{ is even} \\ 0, \text{ if } n \text{ is odd} \end{cases}$.

SOLUTION: Part (b): The generating function of the sequence is $G(x) = 1 + 0 \cdot x + 1 \cdot x^2 + 0 \cdot x^3 + \cdots = 1 + x^2 + x^4 + x^6 \cdots$. If we make the substitution $x \mapsto x^2$ in (8) (which will be a valid equation if $|x| < 1$), we arrive at:

$$\frac{1}{1-x^2} = 1 + x^2 + x^4 + \cdots = \sum_{n=0}^{\infty} x^{2n},$$

which gives an explicit formula for the generating function at hand.

Arithmetic of Generating Functions

Polynomials, i.e., functions of the form

$$a_0 + a_1x + a_2x^2 + \cdots + a_dx^d = \sum_{n=0}^{d} a_nx^n,$$

are determined by a <u>finite</u> sequence of coefficients $a_0, a_1, a_2, \cdots, a_d$ (which we assume to be real numbers). They are the generating functions for their sequence of coefficients. They serve as good motivators for the arithmetic of more general formal power series (i.e., generating functions). When we write down a polynomial, unless all of its coefficients are zero (the zero polynomial) in the representation above, we may always assume that the *leading coefficient*, a_d, is nonzero. In this case the polynomial is said to have *degree d* (the highest power that appears with a nonzero coefficient).

Since polynomials (finite power series) define functions on the entire domain of real numbers, these generating functions can be identified with the functions that they represent.

Polynomials can be added/subtracted term-by-term, and two polynomials can be multiplied in the usual fashion that is taught in basic algebra using the rules $x^n x^m = x^{n+m}$ and $x^0 = 1$. For example, to multiply two general third degree polynomials,

$$(a_0 + a_1x + a_2x^2 + a_3x^3) \cdot (b_0 + b_1x + b_2x^2 + b_3x^3),$$

we could start off as follows:

$$(a_0b_0) + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_1b_1 + a_2b_0)x^2 + (a_0b_3 + a_1b_2 + a_2b_1 + a_3b_0)x^3 + \cdots.$$

For higher powers (up to six), the coefficients are composed of fewer and fewer terms, and thus the computations are simplified. The degree four (next) term is:

$$(a_1b_3 + a_2b_2 + a_3b_1)x^4.$$

If we adopt the convention indicated above that unlisted coefficients are assumed to be zero, we could rewrite this degree four term to fit the general pattern of the first three:

$$(a_0b_4 + a_1b_3 + a_2b_2 + a_3b_1 + a_4b_0)x^4.$$

This latter approach, although more complicated for polynomials, leads to the definition of multiplication of formal power series that is included in Part (c) of the following definition:

DEFINITION 5.5: (*Arithmetic of Generating Functions/Formal Power Series*):
Suppose we have two generating functions $F(x) = \sum_{n=0}^{\infty} a_nx^n$, $G(x) = \sum_{n=0}^{\infty} b_nx^n$.

(a) The **sum** of these generating functions is

$$F(x)+G(x)=\sum_{n=0}^{\infty}(a_n+b_n)x^n.$$

(b) The **difference** of these generating functions is

$$F(x)-G(x)=\sum_{n=0}^{\infty}(a_n-b_n)x^n.$$

(c) The **product** of these generating functions is

$$F(x)\cdot G(x)=\sum_{n=0}^{\infty}(a_0b_n+a_1b_{n-1}+a_2b_{n-2}+\cdots+a_nb_0)x^n=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}a_kb_{n-k}\right)x^n.$$

Calculus books prove that if the series for $F(x)$ and $G(x)$ both have positive radii of convergence, then the formally defined series above really do correspond to the generating functions for the sum, difference, and product of $F(x)$ and $G(x)$, and the equations will be valid if $|x|$ is less than the minimum of the two radii of convergence.

EXAMPLE 5.19: Find a formula for the *n*th coefficient a_n of the sequence whose generating function is given by $e^x/(1-x)$.

SOLUTION: We will use Part (c) of Definition 5.5, along with equations (8) and (9):

$$\frac{e^x}{1-x}=e^x\cdot\frac{1}{1-x}=\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)\cdot\left(1+x+x^2+x^3+\cdots\right)$$
$$=1\cdot1+(1\cdot1+1\cdot1)x+\left(1\cdot1+1\cdot1+\frac{1}{2!}\cdot1\right)x^2+\left(1\cdot1+1\cdot1+\frac{1}{2!}\cdot1+\frac{1}{3!}\cdot1\right)x^3+\cdots.$$

The pattern has quickly revealed itself, and we may write $e^x/(1-x)=\sum_{n=0}^{\infty}a_nx^n$, where $a_n=\frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}=\sum_{k=0}^{n}\frac{1}{k!}$.

From Definition 5.5, it is clear that the generating function 0 (corresponding to the infinite sequence of zeros: $a_n=0$) serves as the additive identity, and the generating function 1, corresponding to the sequence $a_1=1,\ a_n=0,$ if $n\neq0,$ serves as the multiplicative identity. Put differently, if $F(x)$ is any generating function, then $F(x)+0=F(x)$ and $F(x)\cdot1=F(x)$.

EXERCISE FOR THE READER 5.17: Obtain closed-form expressions for the following generating functions.

(a) $x^2+x^3+x^4+\cdots=\sum_{n=2}^{\infty}x^n$ (b) $(1-x)\left(1+2x+\frac{4x^2}{2!}+\frac{8x^3}{3!}+\cdots\right)$.

The Generalized Binomial Theorem

The binomial generating function that was obtained in Example 5.17:

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k,$$

has a very useful extension to non-integer powers. In order to state the result, it is convenient to generalize the definition of the binomial coefficients $\binom{n}{k}$ for non-integer values of n. The following definition does this using a formula that is equivalent to the formula (4) for (ordinary) binomial coefficients in case n is a nonnegative integer.

DEFINITION 5.6: If a is a real number and k is a nonnegative integer, we define the generalized binomial coefficient $\binom{a}{k}$ by the formula:

$$\binom{a}{k} = \begin{cases} \dfrac{a(a-1)(a-2)\cdots(a-k+1)}{k!}, & \text{if } k > 0, \\ 1, & \text{if } k = 0. \end{cases}$$

EXAMPLE 5.20: Compute the generalized binomial coefficient $\binom{3/2}{4}$.

SOLUTION: Substituting a = 3/2 and k = 4 into the formula of the above definition gives:

$$\binom{3/2}{4} = \frac{(3/2)(1/2)(-1/2)(-3/2)}{4!} = \frac{3}{128}.$$

EXERCISE FOR THE READER 5.18: If a is a negative integer: $a=-n$, show that the generalized binomial coefficient $\binom{a}{k}$ can be expressed using ordinary binomial coefficients as follows: $\binom{a}{k} = (-1)^k C(n+k-1, k)$.

We are now ready to state the generalized binomial theorem that provides a generating function power series expansion for $(1+x)^a$, where a is any real number. Unless a is a nonnegative integer, the power series will be an infinite series.

THEOREM 5.11: (*The Generalized Binomial Theorem*) If a is any real number, and x is a real number with $|x|<1$, then

$$(1+x)^a = 1 + \binom{a}{1}x + \binom{a}{2}x^2 + \binom{a}{3}x^3 + \cdots = \sum_{n=0}^{\infty}\binom{a}{n}x^n. \qquad (10)$$

We point out that in case a is a positive integer, then (by Definition 5.6) $\binom{a}{k} = 0$ whenever $k > a$, so the expansion (10) is a finite series and (10) reduces to the ordinary binomial theorem (Theorem 5.7), but in all other cases, $\binom{a}{k} \neq 0$ for all positive integers k, so (10) is really an infinite series.

EXAMPLE 5.21: Apply the generalized binomial theorem to find power series expansion for the following function: $1/(1-x)^2$.

SOLUTION: If we substitute $a \mapsto -2, x \mapsto -x$ into (10), the following expansion that is valid for $|x| < 1$ results:

$$\frac{1}{(1-x)^2} = 1 + \binom{-2}{1}(-x) + \binom{-2}{2}(-x)^2 + \binom{-2}{3}(-x)^3 + \cdots = \sum_{n=0}^{\infty}\binom{-2}{n}(-x)^n.$$

Since $\binom{-2}{n} = \dfrac{-2(-3)(-4)\cdots(-2-n+1)}{1\cdot 2\cdot 3\cdots n} = \dfrac{(-1)^n 2\cdot 3\cdot 4\cdots(n+1)}{1\cdot 2\cdot 3\cdots n} = (-1)^n(n+1)$,

the above expansion becomes:

$$\frac{1}{(1-x)^2} = 1 + -2(-x) + 3(-x)^2 + -4(-x)^3 + \cdots = 1 + 2x + 3x^2 + 4x^3 + \cdots = \sum_{n=0}^{\infty}(n+1)x^n.$$

This expansion could have also been obtained by multiplying that of (8) by itself (using Definition 5.5(c)). Since it is often useful, we record it as numbered equation for future reference (the equality is valid when $|x| < 1$):

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots = \sum_{n=0}^{\infty}(n+1)x^n. \qquad (11)$$

The following more general expansion will often be useful; its justification is similar to the argument in Example 5.21, and is left as the next exercise for the reader. If n is a positive integer and $|x| < 1$, then we have:

$$\begin{aligned}\frac{1}{(1-x)^a} &= 1 + C(a,1)x + C(a+1,2)x^2 + C(a+2,3)x^3 + \cdots \\ &= \sum_{n=0}^{\infty} C(n+a-1, a-1)x^n.\end{aligned} \tag{12}$$

EXERCISE FOR THE READER 5.19: Establish the power series expansion (12).

EXERCISE FOR THE READER 5.20: Determine the power series expansion for the function $\sqrt{1+x/2}$.

Using Generating Functions to Solve Recursive Sequences

Now that we have presented a decent collection of generating functions, it is time to show their usefulness in combinatorics. We will begin by demonstrating some general schemes by which generating functions can be used to solve recursive relations. We begin with an easy example for which an explicit formula can quickly be deduced without much work.

EXAMPLE 5.22: Consider the recursively defined sequence:

$$\begin{cases} a_0 = 1 \\ a_n = 2a_{n-1} + 1 \ (n \geq 1). \end{cases}$$

An explicit formula for this sequence can be easily obtained, for example, by computing the first few terms: $a_1 = 3,\ a_2 = 7,\ a_3 = 15,\ a_4 = 31,$ and discovering the pattern $a_n = 2^{n+1} - 1,$ which can then be established by induction (see also Example 3.6). We will use this example to showcase how generating functions can be used to solve recursive relations. Instead of working with the terms of the sequence, we will consider the generating function for this sequence: $F(x) = \sum_{n=0}^{\infty} a_n x^n,$ and use the given recurrence relation to obtain a closed from expression for this function. We multiply both sides of the recurrence relation by $x^n,$ and then take the formal (infinite) sum of both sides in the range $n \geq 1$ where the recurrence is valid:

$$a_n = 2a_{n-1} + 1 \ (n \geq 1) \Rightarrow \sum_{n=1}^{\infty} a_n x^n = 2\sum_{n=1}^{\infty} a_{n-1} x^n + \sum_{n=1}^{\infty} 1 \cdot x^n.$$

Now let us look at each of these three formal sums and aim to find the corresponding generating functions. The first sum is just

$$\sum_{n=1}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^n - a_0 = F(x) - 1.$$

Relating the second formal sum to $F(x)$ requires a slightly different manipulation:

$$2\sum_{n=1}^{\infty} a_{n-1} x^n = 2x\sum_{n=1}^{\infty} a_{n-1} x^{n-1} = 2x\sum_{n=0}^{\infty} a_n x^n = 2xF(x).$$

The third formal sum is closely related to the expansion (8):

$$\sum_{n=1}^{\infty} x^n = \sum_{n=0}^{\infty} x^n - 1 = \frac{1}{1-x} - 1.$$

Thus, the formal series equation above corresponds to the following equation for the generating function $F(x)$:

$$F(x) - 1 = 2xF(x) + \frac{1}{1-x} - 1 \Rightarrow (1-2x)F(x) = \frac{1}{1-x} \Rightarrow F(x) = \frac{1}{(1-x)(1-2x)}.$$

This generating function determines the entire sequence (a_n). In order to obtain an explicit formula for the terms of the sequence, we use the *partial fractions method* [8] of algebra to expand the expression on the right side:

$$\frac{1}{(1-x)(1-2x)} = \frac{A}{(1-x)} + \frac{B}{(1-2x)}.$$

To determine the constants A and B on the right side, we first multiply both sides of the equation by the denominator on the left to obtain:

$$1 = A(1-2x) + B(1-x).$$

If we substitute $x = 1$ into this equation, we obtain $A = -1$, and substituting $x = 1/2$ produces $B = 2$. The original equation now gives:

$$F(x) = \frac{-1}{(1-x)} + \frac{2}{(1-2x)}.$$

Each of the terms on the right can easily be expanded using (8):

---

[8] The partial fractions expansion applies to any quotient of polynomials $P(x)/Q(x)$, where the degree of the numerator is less than that of the denominator, and the denominator $Q(x)$ is factored into powers of distinct linear factors $(x-a)^k$, where a is a complex number. Each factor of $Q(x)$ of form $(x-a)^k$ gives rise to a sum of terms of form: $\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_2}{(x-a)^k}$, where $A_1, A_2, \cdots, A_k$ are constants, in the partial fraction expansion. Each such term can be expanded into a power series using (12). If $P(x)$ were to have higher degree than $Q(x)$, a long division of polynomials could be used to rewrite $P(x)/Q(x)$ as a sum of a polynomial and $R(x)/Q(x)$, where the remainder $R(x)$ has smaller degree than $Q(x)$. Technical note: Although the examples and exercises given in this book will involve only real numbers, the methods that we develop still work in cases of complex numbers (because (12) remains valid if x is a complex number of modulus less than 1).

$$F(x)=\frac{1/2}{(1-x)}+\frac{2}{(1-2x)}=-1\sum_{n=0}^{\infty}x^n+2\sum_{n=0}^{\infty}(2x)^n=\sum_{n=0}^{\infty}(2^{n+1}-1)x^n.$$

Thus we have found that $a_n = 2^{n+1}-1$.

Although the heavy machinery developed in this example was not really needed, the generating function method can be used in the same fashion to solve recurrences that are not so easily solved by other means. This is one of the beauties of the generating function method.

EXERCISE FOR THE READER 5.21: Use the generating function method of the previous example to solve the recurrence: $\begin{cases} a_0 = 1 \\ a_n = 3a_{n-1} - 1 \; (n \ge 1). \end{cases}$

Our next example involves a recurrence whose solution is not so amenable to discovery as in the preceding example.

EXAMPLE 5.23: Use the method of generating functions (as developed in Example 5.22) to determine an explicit formula for the following recursively defined sequence:

$$\begin{cases} a_1 = 4 \\ a_n = 3a_{n-1} + 2n - 1 \; (n \ge 2). \end{cases}$$

SOLUTION: In order to facilitate the use of generating functions, we extend the definition of the sequence to define a_0 in a way that will make the recursion formula valid for n = 1 (and hence for $n \ge 1$). If we use the recursion formula with n = 1: $a_1 = 3a_0 + 2\cdot 1 - 1,$ substitute $a_1 = 4,$ we obtain $a_0 = 1.$

Following the method of the preceding example, we let $F(x)=\sum_{n=0}^{\infty} a_n x^n$ be the generating function for the given sequence, multiply both sides of the recurrence relation by x^n, and then take the formal (infinite) sum of both sides in the range $n \ge 1$ (where the recurrence is valid):

$$a_n = 3a_{n-1} + 2n - 1 \;\; (n \ge 1) \Rightarrow \sum_{n=1}^{\infty} a_n x^n = 3\sum_{n=1}^{\infty} a_{n-1}x^n + 2\sum_{n=1}^{\infty} nx^n - \sum_{n=1}^{\infty} x^n.$$

Three out of the four sums on the right can be converted to closed-form expressions as in the preceding example. The second sum on the right can be converted using the expansion (11) as follows:

$$2\sum_{n=1}^{\infty} nx^n = 2x\sum_{n=1}^{\infty} nx^{n-1} = 2x\sum_{n=0}^{\infty}(n+1)x^n = \frac{2x}{(1-x)^2}.$$

Hence, the preceding equation involving four power series transforms into:

$$F(x) - 1 = 3xF(x) + \frac{2x}{(1-x)^2} - \frac{1}{1-x} + 1.$$

Solving this equation for $F(x)$ and converting to a single term gives us:

$$F(x) = \frac{2x^2 - x + 1}{(1-3x)(1-x)^2}.$$

The partial fractions expansion will have the form:

$$F(x) = \frac{2x^2 - x + 1}{(1-3x)(1-x)^2} = \frac{A}{1-3x} + \frac{B}{1-x} + \frac{C}{(1-x)^2}.$$

To determine the three constants A, B, C, we first clear out all denominators:

$$2x^2 - x + 1 = A(1-x)^2 + B(1-x)(1-3x) + C(1-3x).$$

Substituting $x = 1$ yields $C = -1$. Substituting $x = 1/3$ yields $A = 2$, and finally substituting $x = 0$ (or any third number) yields $B = 0$. Thus we have determined the partial fractions expansion of the generating function to be:

$$F(x) = \frac{2}{1-3x} - \frac{1}{(1-x)^2}.$$

Applying the expansions (8) and (11) allows us to determine the corresponding single power series expansion:

$$F(x) = \frac{2}{1-3x} - \frac{1}{(1-x)^2} = 2\sum_{n=0}^{\infty}(3x)^n - \sum_{n=0}^{\infty}(n+1)x^n = \sum_{n=0}^{\infty}(2 \cdot 3^n - n - 1)x^n.$$

We have thus arrived at the following closed formula for the given recursively defined sequence: $a_n = 2 \cdot 3^n - n - 1$.

EXERCISE FOR THE READER 5.22: Use the generating function method to solve the recurrence: $\begin{cases} a_0 = 1 \\ a_n = 2a_{n-1} + 3^n \ (n \geq 1). \end{cases}$

Whereas the methods of Section 3.2 (see Theorems 3.7, 3.8, and 3.9, and the examples given) could have also been applied to solve the recurrences of the preceding example, our next example is not amenable to the theory of Section 3.2, since the formula for the nth term involves all previous terms (rather than a fixed number of them).

EXAMPLE 5.24: Obtain an explicit formula for the following recursively defined sequence:

$$\begin{cases} a_0 = 1 \\ \binom{n+4}{4} = \sum_{k=0}^{n} a_k a_{n-k} \ (n \ge 1). \end{cases}$$

SOLUTION: We first note that the recurrence equation $\binom{n+4}{4} = \sum_{k=0}^{n} a_k a_{n-k}$ remains valid when $n = 0$, and hence is valid for all nonnegative integers n. If, as usual, we let $F(x) = \sum_{n=0}^{\infty} a_n x^n$ be the generating function for the given sequence, we first observe from Definition 5.5(c) that sequence formed by the right side of the recursion formula has the generating function $F(x)^2$. Also, from (12) we see that the coefficients $\binom{n+4}{4}$ have the generating function $\frac{1}{(1-x)^5}$. It follows that if we multiply both sides of the recurrence relation by x^n, and then take the formal (infinite) sum of both sides in the range $n \ge 0$ (where the recurrence is valid) the resulting equation $\sum_{n=0}^{\infty} \binom{n+4}{4} x^n = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k a_{n-k} \right) x^n$ corresponds to the equation $(1-x)^{-5} = F(x)^2$, from which it follows that $F(x) = (1-x)^{-5/2}$. We may apply the generalized binomial Theorem 5.11 (formula (10)) to obtain the expansion of this generating function:

$$F(x) = 1 + \binom{-5/2}{1}(-x) + \binom{-5/2}{2}(-x)^2 + \binom{-5/2}{3}(-x)^3 + \cdots = \sum_{n=0}^{\infty} \binom{-5/2}{n} (-x)^n.$$

But since $\binom{-5/2}{n} = \frac{(-5/2)\cdot(-7/2)\cdots(-2n-3/2)}{n!} = (-1)^n \frac{5\cdot 7\cdots(2n+3)}{2^n n!}$, we obtain the desired explicit formula $a_n = \frac{5\cdot 7\cdots(2n+3)}{2^n n!}$.

EXERCISE FOR THE READER 5.23: Use the generating function method to derive the explicit formula given in equation (4) of Chapter 3 for the famous Fibonacci sequence (first introduced in Example 3.7) that is recursively defined by: $\begin{cases} f_1 = 1, f_2 = 1, \\ f_n = f_{n-1} + f_{n-2} \ (n \ge 3). \end{cases}$

Using Generating Functions in Counting Problems

Generating functions can be used for a variety of counting problems. The hand computations that arise with such methods are typically quite laborious, so it is most convenient to make use of computers for such tasks. The computer implementation material at the end of this chapter contains some useful information in this regard.

In order to motivate the concepts, we begin by redoing the simple Example 5.16, which was used in Section 5.2 to motivate a general partitioning argument. This example will again serve as a motivating example for the application of generating functions to counting problems.

EXAMPLE 5.25: (*Motivating Example for Counting Techniques Using Generating Functions*) Joey has five identical chocolate bars that he plans to give to his three cousins, Abby, Billy, and Christy. Use generating functions to determine the number of different ways that Joey can distribute these bars among his cousins.

SOLUTION: Each cousin can receive from zero to five bars, and this gives rise to the generating polynomial $1(=x^0)+x+x^2+x^3+x^4+x^5$. The exponent represents the number of candy bars the particular cousin receives; since there are no differences on how Joey can distribute the bars to his three cousins, the three generating functions are the same.

We claim that the number of solutions to the problem is the coefficient of x^5 in the product of the three generating polynomials: $(1+x+x^2+x^3+x^4+x^5)^3$, the latter being the generating function for the problem. The reason is that the x^5 term in the product is the sum of all terms of the form $x^A x^B x^C$, where x^A is taken from the first factor (and A represents the number of bars Alice receives), x^B is taken from the second factor (corresponding to Billy receiving B bars), x^C is taken from the third factor, and $A+B+C=5$. There is thus a one-to-one correspondence between the ways that Joey can distribute the five bars among his three cousins, and the terms $x^A x^B x^C$ that arise in expanding the product of the three (in this case identical) generating polynomials of the three cousins. A computation (best done on a computer) shows the coefficient of x^5 in the expansion of $(1+x+x^2+x^3+x^4+x^5)^3$ to be 21, in agreement with the solution to Example 5.16.

The generating function approach to counting is much more versatile than some of the specialized techniques introduced in Section 5.2. The next example is a variation of the previous one that is not so clearly solvable using the techniques of

Section 5.2, but is easily solved by the same technique introduced in the solution of Example 5.25.

EXAMPLE 5.26: Joey has five identical chocolate bars that he plans to give to his three cousins, Abby, Billy, and Christy. Use generating functions to determine the number of different ways that Joey can distribute these bars to his cousins under the following constraints: Abby must get at least one bar and Billy must get an even number of bars.

SOLUTION: There are no constraints on the number of bars that Christy can receive so her generating function is exactly as it was in Example 5.25: $F_C(x) = 1(= x^0) + x + x^2 + x^3 + x^4 + x^5$. Since Abby must receive at least one bar, her generating function is the same but without the $1(= x^0)$ term (since we need to omit the option of giving her zero bars): $F_A(x) = x + x^2 + x^3 + x^4 + x^5$. Finally, Billy's generating function is obtained from Christy's by removing all of the odd powers of x: $F_B(x) = 1(= x^0) + x^2 + x^4$. The generating function for the whole problem is the product of these three:

$$F_A(x) \cdot F_B(x) \cdot F_C(x) = (x + x^2 + x^3 + x^4 + x^5) \cdot (1 + x^2 + x^4) \cdot (1 + x + x^2 + x^3 + x^4 + x^5).$$

The number of ways in which Joey can distribute the five bars subject to the given constraints is the coefficient of x^5 in this product. This coefficient can be easily computed with the aid of a computer (or with a hand computation), and is 9.

NOTE: The generating functions for such counting problems actually contain much more information than is typically used. In Example 5.26, the expanded form of the generating function is:

$$\begin{aligned} F_A(x) \cdot F_B(x) \cdot F_C(x) = x + 2x^2 + 4x^3 + 6x^4 + 9x^5 + 11x^6 + 12x^7 \\ + 12x^8 + 11x^9 + 9x^{10} + 6x^{11} + 4x^{12} + 2x^{13} + x^{14}. \end{aligned}$$

Each coefficient of a certain power x^n in this expansion gives the number of ways that Joey could give out n bars to his three cousins subject to the constraints represented by the generating functions (at most 5 bars can be given to any cousin since x^5 is the highest power appearing in each cousin's generating function). So for example, the coefficient of x is 1, corresponding to the fact that there is only one way for Joey to give out just one bar to his three cousins, due to the constraint that Abby must get at least one bar: Abby: 1, Billy: 0, Christy: 0. Similarly, the coefficient of x^2 being 2 corresponds to the fact that the following are the only ways that Joey could distribute two bars to his cousins under the specified constraints: (i) Abby: 1, Billy: 0, Christy: 1, or (ii) Abby: 2, Billy: 0, Christy: 0.

EXERCISE FOR THE READER 5.24: A winner of a contest is allowed to (blindly) draw (and keep) exactly four bills from an urn that contains ten $1 bills, five \$10 bills, and two \$100 bills. How many different combinations of bills could be chosen? What if instead of four bills, six bills are chosen?

Note that in both of the above examples, we could have added higher powers of x: $x^6, x^7, \cdots$ to each of the three generating functions (Abby's, Billy's, and Christy's) because when we looked for the coefficient of x^5, such higher degree terms would not contribute to this lower degree term. Thus, we could have even used infinite series for these generating functions. Since we do have explicit closed formulas for an assortment of infinite power series, it is sometimes convenient to use infinite power series for generating functions in counting problems.

EXAMPLE 5.27: Use generating functions to give another proof of Theorem 5.10: (*Distribution of identical objects to different places*) The number of ways to distribute n identical objects to d different (distinguishable) places is given by $\binom{n+(d-1)}{d-1}$.

SOLUTION: Although the first idea for the generating function for the number of objects that are placed in the ith place $(1 \le i \le d)$ would be $1 + x + x^2 + \cdots + x^n$ (since there are only n objects in totality that can be placed), it will be more convenient to use the infinite power series: $1 + x + x^2 + \cdots$. The resulting generating function for the whole counting problem is just the product of these d identical individual generating functions $(1 + x + x^2 + \cdots)^d = 1/(1-x)^d$. The number of ways to distribute n identical objects to d different (distinguishable) places is then the coefficient of x^n in this expansion. By (12) (with $a = d$) this coefficient is $C(n+d-1, d-1)$, and the proof is complete.

EXERCISES 5.3:

NOTE: As pointed out in the section proper, calculating coefficients of generating functions is sometimes not feasible without the aid of a computer. In cases where such situations arise in the exercises below, readers who do not have access to an appropriate computing system might choose to pass up hand computations of such coefficients. While symbolic and/or computer algebra systems are very well suited for the polynomial manipulations needed in such calculations, the computer implementation material at the end of this section will provide details on performing such computations on any standard computing platform.

1. Write down the generating function for each of the following sequences:
   (a) $a_n = (-1)^n \ (n = 0,1,2,\cdots)$ (b) $a_0 = 6,\ a_2 = 4,\ a_4 = 2,\ a_6 = 1$

2. Write down the generating function for each of the following sequences:

(a) $a_n = 2^{n+3}\ (n = 0,1,2,\cdots)$

(b) $a_1 = 2,\ a_2 = 4,\ a_3 = 2,\ a_8 = 4$

3. For each of the following finite sequences, (i) write down the (polynomial) generating function, and (ii) if possible use either the binomial theorem (Theorem 5.7) or the formula for finite geometric series (Theorem 3.5) to express the function as a closed-form expression.

(a) $a_n = (-1)^n\ (n = 0,1,2,\cdots,10)$

(b) $a_n = C(10,n)\ (n = 0,1,2,\cdots,10)$

(c) $a_0 = 20,\ a_1 = -40,\ a_2 = 80,\ a_3 = -160,\ a_4 = 320$

(d) $a_n = C(5,n-2)\ (n = 2,3,4,5,6,7)$

4. For each of the following finite sequences, (i) write down the (polynomial) generating function, and (ii) if possible use either the binomial theorem (Theorem 5.7) or the formula for finite geometric series (Theorem 3.5) to express the function as a closed-form expression.

(a) $a_0 = 6,\ a_1 = 3,\ a_2 = 3/2,\ a_3 = 3/4,\ a_4 = 3/8$

(b) $a_n = C(10,10-n)\ (n = 0,1,2,\cdots,10)$

(c) $a_0 = 10,\ a_1 = 30,\ a_2 = 90,\ a_3 = 270,\ a_4 = 810$

(d) $a_n = 2^n C(4,n)\ (n = 0,1,2,3,4)$

5. Obtain closed-form algebraic expressions for the generating functions defined by the following infinite sequences:

(a) $a_n = (-1)^n\ (n = 0,1,2,\cdots)$

(b) $a_n = (-2)^n / n!\ (n = 0,1,2,\cdots)$

(c) $a_1 = 5,\ a_n = (-1)^n\ (n = 0,2,3,\cdots)$

(d) $a_n = 1/(n+2)!\ (n = 0,1,2,3,\cdots)$

6. Obtain closed-form algebraic expressions for the generating functions defined by the following infinite sequences:

(a) $a_n = \begin{cases} 0, & \text{if } n = 0,2,4\cdots \\ -2, & \text{if } n = 1,3,5\cdots \end{cases}$

(b) $a_n = \begin{cases} (-1)^{n/2}, & \text{if } n = 0,2,4\cdots \\ 0, & \text{if } n = 1,3,5\cdots \end{cases}$

(c) $a_n = \begin{cases} 2, & \text{if } n = 0,1,2 \\ 0, & \text{if } n = 2,4,6\cdots \\ -2, & \text{if } n = 3,5,7\cdots \end{cases}$

(d) $a_0 = a_1 = 1, a_n = 1/(n-2)!\ (n = 2,3,4\cdots)$

(e) $a_n = 3^{n-1}/(n+2)!\ (n = 0,1,2,3,\cdots)$

7. Determine the sequence corresponding to each of the following closed-form expressions for generating functions.

(a) $x^3(x+5)^4$ (b) $(1-x)^3 - x^5$ (c) $1/(1+3x)$

(d) $1/(1+x) - x/(1-2x)$ (e) $1/[(1+x)(1-2x)]$ (f) $e^{2x}(1-x^2)$

8. Determine the sequence corresponding to each of the following closed-form expressions for generating functions.

(a) $x^2(1-2x)^4$ (b) $(x+3)^4 + x^3$ (c) $x/(1+x^2)$

(d) $1/(1+x^2) - x/(1-2x)$ (e) $x/[(1+x^2)(1-2x)]$ (f) $x^2(1+x)e^{-x}$

9. Determine the coefficient of x^8 in each of the following expansions.
   (a) $(1+x+x^2+x^3)(1+x^2+x^4+x^6)(1+x^3+x^6+x^9)$
   (b) $(1+x+x^2+x^3)^3$
   (c) $(1+2x+3x^2+4x^3+\cdots)(1-x^2+x^4-x^6+\cdots)$
   (d) $(1+x+x^2+x^3+\cdots)+(x+x^2+x^3+x^4+\cdots)^2+(x^2+x^3+x^4+\cdots)^3+\cdots$

10. Determine the coefficient of x^9 in each of the expansions of Exercise 9.

11. Suppose that the generating function of a certain sequence $\{a_n\}_{n=0}^{\infty}$ has a closed-form expression $F(x)$.
    (a) What is the sequence that has $x^3F(x)$ as its generating function?
    (b) What is the sequence that has $(1-x)F(x)$ as its generating function?
    (c) What is the sequence that has $F(x)/(1-x)$ as its generating function?

12. Suppose that the generating function of a certain sequence $\{a_n\}_{n=0}^{\infty}$ has a closed-form expression $F(x)$.
    (a) What is the sequence that has $2x^2F(x)$ as its generating function?
    (b) What is the sequence that has $(1+x)F(x)$ as its generating function?
    (c) What is the sequence that has $F(x)/(2+x^2)$ as its generating function?

13. Evaluate each of the following generalized binomial coefficients:
    (a) $\binom{-6}{3}$ (b) $\binom{3.5}{5}$

14. Evaluate each of the following generalized binomial coefficients:
    (a) $\binom{-1/2}{4}$ (b) $\binom{0.9}{5}$

15. (a) Determine the power series expansion for the following generating function: $1/\sqrt{x+1}$.
    (b) For each of the following closed-form generating functions, determine the first four terms of the corresponding power series expansion:
    (i) $(1+x)^{3.5}$ (ii) $e^x/\sqrt{1+x}$

16. (a) Determine the power series expansion for the following generating function: $\sqrt{1+x}$.
    (b) For each of the following two closed-form generating functions, determine the first four terms of the corresponding power series expansion:
    (i) $(3-5x)^{2.5}$ (ii) $e^{2x}\sqrt{1+x}$

17. Use the generating function method to find explicit formulas for each of the following recursively defined sequences:
    (a) $\begin{cases} a_0 = 3 \\ a_n = 2a_{n-1}+5 \ (n \ge 1) \end{cases}$ (b) $\begin{cases} a_2 = 1 \\ a_n = 3a_{n-1}-1 \ (n \ge 3) \end{cases}$ (c) $\begin{cases} a_0 = 1 \\ a_n = 2a_{n-1}+3n \ (n \ge 1) \end{cases}$
    (d) $\begin{cases} a_0 = 1,\ a_1 = 1, \\ a_n = 2a_{n-2}+5 \ (n \ge 2) \end{cases}$ (e) $\begin{cases} a_0 = 1,\ a_1 = 2, \\ a_n = 2a_{n-2}+a_{n-1} \ (n \ge 2) \end{cases}$

18. Use the generating function method to find explicit formulas for each of the following recursively defined sequences:

(a) $\begin{cases} a_0 = 1 \\ a_n = 4a_{n-1} - 1 \ (n \geq 1) \end{cases}$ (b) $\begin{cases} a_{10} = 4 \\ a_n = 2a_{n-1} + 2 \ (n \geq 11) \end{cases}$ (c) $\begin{cases} a_0 = 3 \\ a_n = 3a_{n-1} + 2n \ (n \geq 1) \end{cases}$

(d) $\begin{cases} a_0 = 1,\ a_1 = 1, \\ a_n = 3a_{n-2} + 2 \ (n \geq 2) \end{cases}$ (e) $\begin{cases} a_0 = 1,\ a_1 = 1, \\ a_n = 2a_{n-2} + 3a_{n-1} - 2 \ (n \geq 2) \end{cases}$

19. Use the generating function method to find explicit formulas for each of the following recursively defined sequences:

(a) $\begin{cases} a_0 = 1 \\ 1 = a_n + 2a_{n-1} + 3a_{n-2} + \cdots + na_1 + (n+1)a_0 \ \ (n \geq 1) \end{cases}$

(b) $\begin{cases} a_0 = 2 \\ n = a_n + 2a_{n-1} + 3a_{n-2} + \cdots + na_1 + (n+1)a_0 \ \ (n \geq 1) \end{cases}$

20. Use the generating function method to find explicit formulas for each of the following recursively defined sequences:

(a) $\begin{cases} a_0 = 4 \\ 1 = a_n + a_{n-2} + a_{n-4} + \cdots \ (n \geq 1) \end{cases}$ (b) $\begin{cases} a_0 = 4 \\ n+1 = a_n + a_{n-2} + a_{n-4} + \cdots \ (n \geq 1) \end{cases}$

For each counting problem in Exercises 21–26 below, do the following: (i) Write down a generating function for the problem, and indicate which coefficient will be the answer to the problem. (ii) Determine this coefficient, and hence the answer to the problem.
NOTE: As demonstrated in the section proper, generating functions for counting problems need not be unique.

21. (a) In how many ways can seven bills among \$1, \$5, or \$10 bills be distributed to Jimmy?
(b) How many combinations of rainy and sunny days can there be over 1 week (ignore the order of the days)?
(c) In how many combinations can 10 drinks be ordered from the choices of beers, glasses of wine, or martinis?

22. (a) In how many ways can one choose six coins from a tin containing four pennies and six dimes?
(b) How many combinations of eight stamps can be formed using 1¢, 3¢, or 5¢ stamps?
(c) How many ways can one place 10 toppings on a pizza from among the following topping choices: pepperoni, artichoke, chicken, onions, cheese, and bell peppers (so multiple toppings must be selected, e.g., 4 artichoke, 4 chicken, and 2 bell pepper toppings; ignore the order of the toppings)?

23. (a) In how many ways can seven bills among \$1, \$5, and \$10 bills be distributed to Jimmy if he must get at least one \$10 bill, and an odd number of \$5 bills?
(b) How many combinations of rainy and sunny days can there be over 1 week (ignore the order of the days) if there are an odd number of rainy days and at most 5 sunny days?
(c) In how many combinations can 10 drinks be ordered from the choices of beers, glasses of wine, or martinis, if there must be at least 2 martinis and there cannot be only one glass of wine?

24. (a) How many ways can one choose six coins from a tin containing four pennies and six dimes if an odd number of pennies were selected, and at least two dimes were selected?
(b) How many combinations of eight stamps can be formed using 1¢, 3¢, or 5¢ stamps, if at least two 5¢ stamps are used, and at most five 1¢ stamps are used?
(c) How many ways can one place 10 toppings on a pizza among the following topping choices: pepperoni, artichoke, chicken, onions, cheese, and bell peppers (so multiple toppings must be selected, e.g., 4 artichoke, 4 chicken, and 2 bell pepper toppings) if at least one topping must be artichoke, and at most five toppings are cheese?

25. In how many ways can 6 pieces of fruit be chosen from a basket that contains five (identical) apples, six oranges, two pineapples, and three bananas if at most one pineapple can be chosen, and if one banana is chosen then all must be chosen?

26. (a) In how many ways can a \$100 bill be exchanged into smaller bills using any of the following denominations: \$1, \$5, \$10, \$20, \$50?
(b) How would the answer in Part (a) change if we could also use \$2 bills?
(c) In how many ways can a \$1 bill be changed using coins of the following values: 50¢, 25¢, 10¢, 5¢, and/or 1¢?

NOTE: (***Partitions of Integers***) A **partition** of a positive integer *n* is an (unordered) list of positive integers whose sum is *n*. The number of partitions of *n* is called the **partition function** and is denoted as $p(n)$. For example, the partitions of 4 are easily seen to be:

$$4,\ 3+1,\ 2+2,\ 2+1+1,\ \text{and } 1+1+1+1$$

so that $p(4)=5$.

The **parts** of a partition are simply the integers that appear in it.
For each positive integer *m*, we define the related functions:

$p_m(n)$ = the number of partitions of *n* each of whose parts is at most *m*,

$q_m(n)$ = the number of partitions of *n* that consist of at most *m* parts.

For example, $p_2(4)=3$ since three of the above listed partitions of four have all parts being at most two; also $q_2(4)=3$ since exactly three of the above listed partitions of four have at most two Parts (4, 3 + 1, and 2 + 2). Since any part of a partition of *n* can be at most *n* and since there can be at most *n* parts, it follows that $p_m(n)=p(n)=q_m(n)$ whenever $m \ge n$. Although it is not immediately obvious, it turns out that $p_m(n)=q_m(n)$, as will be shown in Exercise 32.

Partitions are important in number theory and certain combinatorial problems. Although there is no efficient algorithm for computing the partition function, generating functions can be used in the study of partitions. This will be the subject of the Exercises 27–32.

27. (a) Show that a generating function for the sequence $\{p_m(n)\}_{n=1}^{\infty}$ (the number of partitions of *n* whose parts are each at most *m*) is given by $G_m(x)=\dfrac{1}{(1-x)(1-x^2)(1-x^3)\cdots(1-x^m)}$, and thus, when $m \ge n$ this also serves as a generating function of the partition function $p(n)$. Use this generating function to compute $p(n)$, for *n* = 4, 5, 6, and 8.
(b) Determine the value of *m* and the coefficient of $G_m(x)$ that will give the answer to the following counting problem, and then obtain the answer: *In how many ways can a postage of 15¢ be made using stamps of values 1¢, 2¢, 3¢, 4¢, and/or 5¢?*
NOTE: According to the fact mentioned in the preceding note, the answer in Part (b) will also be the answer to the following problem: *In how many ways can a 15¢ postage be made using at most five stamps of values 1¢, 2¢, 3¢, 4¢, ...,14¢, or 15¢?*
Suggestion: For both Parts (a) and (b), use the expansions $1/(1-x^k)=1+x^k+x^{2k}+x^{3k}+\cdots$ (which follow from (8), for any positive integer *k*).

28. (a) Show that the function $G_m(x)=\dfrac{1}{(1-x)(1-x^2)(1-x^3)\cdots}$ is a generating function of the partition function $p(n)$.
(b) Let $p_O(n)$ denote the number of partitions of *n* into odd integer Parts (i.e, as a sum of odd integers). Since 1 + 1 + 1 + 1 and 3 + 1 are the only such partitions of 4, we have $p_O(4)=2$.

Find a generating function for $p_O(n)$, and then use it to compute $p_O(n)$ for n = 4, 5, 6, 7, and 10.
NOTE: Although the generating function defined in Part (a) involves an infinite product, each resulting term in the expansion (with the exception of 1, which is the product of an infinite number of 1's) involves only a finite product and there are only finitely many terms associated with each power of x.

29. (a) Let $p_D(n)$ denote the number of partitions of n into distinct Parts (i.e, no two parts of the partition are the same number). Since 4 and 3 + 1 are the only such partitions of 4, we have $p_D(4) = 2$. Find a generating function for $p_D(n)$.

(b) Use your generating function of Part (a) to compute $p_D(n)$ for n = 4, 5, 6, 7, and 10.

30. (*Euler's Theorem*) Show that the function $p_O(n)$ of Exercise 28 is the same as the function $p_D(n)$ of Exercise 29 by showing they have the same generating function.

31. Use generating functions to show that every positive integer can be uniquely expressed as a sum of distinct powers of 2.

32. (*Star Diagram and a Proof that* $p_m(n) = q_m(n)$) Fill in the details of the following outline of a proof of the fact, mentioned above, that the number of partitions of a positive integer n into at most m parts is the same as the number of partitions of n into parts of size at most m, i.e., that $p_m(n) = q_m(n)$: Any partition of n can be represented by its *star diagram* that consists of n stars (or asterisks) grouped in rows corresponding to the parts of the partitions. For example, the partition 5 + 4 + 2 of 11 has the following star diagram:

```
* * * * *
* * * *
* *
```

Each star diagram has a conjugate star diagram obtained viewing the columns as the parts rather than the rows (i.e., transposing rows and columns), and this gives rise to a *conjugate partition*. The conjugate partition of the preceding is 11 = 3 + 3 + 2 + 2 + 1, which has the following star diagram:

```
* * *
* * *
* *
* *
*
```

Show that the conjugate operation bijectively maps all partitions of n into m parts into all partitions of n into parts of size at most m, thus proving that $p_m(n) = q_m(n)$.

NOTE: (*Sicherman Dice*) The next two exercises introduce an interesting problem relating to dice, along with a solution to this problem using generating functions. The problem asks whether it is possible to repaint the six faces of a pair of dice with positive integers (with duplications being allowed on the same die) in such a way that is different from the standard die (i.e., the six faces contain the numbers 1 through 6) and that when these modified dice are rolled together, the number of ways that the sum of the two numbers appearing as a given value (between 2 and 12) will be the same as for a pair of standard dice. Such a pair of dice were discovered by Colonel George Sicherman (a computer programmer) and first appeared in the literature in a 1978 *Scientific American* article by Martin Gardner. The next two exercises will use generating functions to find Sicherman's dice, and show that they are unique.

33. (*Rolling Generalized Dice*)

(a) Explain why the function $D(x) = (x + x^2 + x^3 + x^4 + x^5 + x^6)(x + x^2 + x^3 + x^4 + x^5 + x^6)$ serves as a generating function for the problem of counting the number ways that a particular

outcome occurs when two dice are tossed and the outcome is viewed as the sum of the two numbers on the top faces (i.e., an integer between 2 and 12 inclusive). Compute the coefficient of x^4 of this generating function, and show that it is the same as the number of ways that a 4 (total) can be obtained by rolling two dice.
(b) Next, suppose that the six faces of a pair of dice are repainted with numbers that are (not necessarily distinct) positive integers:

Die 1: $a_1 \le a_2 \le a_3 \le a_4 \le a_5 \le a_6$ Die 2: $b_1 \le b_2 \le b_3 \le b_4 \le b_5 \le b_6$

Show that a generating function for the problem of counting the number of ways that these two dice can be tossed and the sum of the numbers appearing on the top faces adding up to a certain number (i.e., an integer between $a_1 + b_1$ and $a_6 + b_6$ inclusive) is given by:

$$(x^{a_1} + x^{a_2} + x^{a_3} + x^{a_4} + x^{a_5} + x^{a_6})(x^{b_1} + x^{b_2} + x^{b_3} + x^{b_4} + x^{b_5} + x^{b_6})$$

Check this result by expanding the function in the very simple case in which all numbers on Die 1 are 1s and all numbers on Die 2 are 2s.

34. (*Sicherman Dice*) (a) Use the generating functions established in the preceding exercise to show that there is only one pair of Sicherman dice, namely those dice with the following face values: 1, 2, 2, 3, 3, 4, and 1, 3, 4, 5, 6, 8.
(b) Verify that the 36 possible outcomes obtained by adding the numbers shown of the Sicherman dice given in Part (a) really do amount to the same number of outcomes for the numbers 2–12 as in the case for ordinary dice.
Suggestion: Use the fact that the generating function for a standard pair of dice (as given in Part (a) of the preceding exercise) factors as follows:

$$(x + x^2 + x^3 + x^4 + x^5 + x^6)^2 = x^2(1+x)^2(1+x+x^2)^2(1-x+x^2)^2.$$

We will also need the fact that polynomials with integer coefficients obey a unique factorization property similar to that of the integers.[9] By equating this generating function to the generating function for a pair of Sicherman dice (from Part (b) of the previous exercise):

$$\begin{aligned}(x^{a_1} + x^{a_2} + x^{a_3} + x^{a_4} + x^{a_5} + x^{a_6})&(x^{b_1} + x^{b_2} + x^{b_3} + x^{b_4} + x^{b_5} + x^{b_6}) \\ &= x^2(1+x)^2(1+x+x^2)^2(1-x+x^2)^2,\end{aligned}$$

it follows (from unique factorization) that the individual generating functions for each of the two dice must satisfy:

$$x^{a_1} + x^{a_2} + x^{a_3} + x^{a_4} + x^{a_5} + x^{a_6} = x^{\alpha_1}(1+x)^{\alpha_2}(1+x+x^2)^{\alpha_3}(1-x+x^2)^{\alpha_4},$$

and

$$x^{b_1} + x^{b_2} + x^{b_3} + x^{b_4} + x^{b_5} + x^{b_6} = x^{\beta_1}(1+x)^{\beta_2}(1+x+x^2)^{\beta_3}(1-x+x^2)^{\beta_4},$$

where α_i, β_i are nonnegative integers that satisfy $\alpha_i + \beta_i = 2$, for $i = 1, 2, 3, 4$. Substitute $x = 0$ into both of the above equations to conclude that $\alpha_1 = 1 = \beta_1$. Substitute $x = 1$ into both of the above equations to conclude that $\alpha_2 = \alpha_3 = 1 = \beta_2 = \beta_3$. Finally, consider the generating function for the pair and show there are now three feasible solutions: (i) $\alpha_4 = 1 = \beta_4$, (ii) $\alpha_4 = 0, \beta_4 = 2$, or (iii) $\alpha_4 = 2, \beta_4 = 0$.

35. (a) Establish the expansion

$$(x + x^2 + \cdots + x^9)(1 + x + x^2 + \cdots + x^9)^5 = x(1-x^9)(1-x^{10})^5(1-x)^{-6}$$

and then show this function is a generating function for the problem of counting the number of

[9] The interested reader can find details on this topic in any good book on abstract algebra, such as [Hun-96]. The analog of the prime numbers in this polynomial factorization theory are so-called (integer coefficient) *irreducible* polynomials: they cannot be further factored into integer coefficient polynomials of smaller degree. Another relevant topic in abstract algebra is the existence of algorithms for performing such integer factorizations of polynomials. Such algorithms are built in to most symbolic/computer algebra systems.

positive six digit integers the sum of whose digits is n.
(b) How many positive six digit numbers have a digit sum of 22?
(c) Find a generating function for the problem of counting the number of k-digit positive integers whose digits sum to n.

36. (a) Find a generating function for the problem of counting the number ofpositive integers between 1 and $10^k - 1$ (inclusive) whose digits sum to n.
(b) How many positive integers in the range 1 to 999,999 have a digit sum of 22?
Suggestion: See Exercise 35(a).

APPENDIX TO SECTION 5.3: APPLICATION TO WEIGHTED DEMOCRACIES

We close this section with an application to weighted voting systems. It will provide an excellent illustration of how generating functions can sometimes be used to produce algorithms that are significantly more efficient than other methods for solving combinatorial problems. The implementation of generating functions for the problem that we discuss is more sophisticated and less transparent than for the other counting problems we have discussed so far; the development is due to J. M. Bilbao, et. al [BFLL-00]. A different and more general method was subsequently discovered by V. Yakuba [Yak-08].

In many democratic voting systems, it is often equitable for different voters to have different powers of vote. For example, at a stockholder meeting, people who own greater numbers of shares are allowed proportionately stronger votes. Similarly, in the European Union, larger nations (like France or the UK) have five times as much voting power as some of the smaller nations (like Luxembourg). Let us first define the voting systems that we will be considering:

DEFINITION 5.7: A **weighted voting system** consists of the following:
A set of N **voters**: $V_1, V_2, \cdots, V_N$, a corresponding set of N **weights**: $w_1, w_2, \cdots, w_N$, which are assumed to be positive integers giving the number of votes controlled by the corresponding voters, and a **quota** q, which is a positive integer satisfying $\sum_{n=1}^N w_i \ge q > (1/2)\sum_{n=1}^N w_i$ equaling the minimum number of votes needed to pass a motion that is being voted on. A weighted voting system with these parameters will be denoted by $[q : w_1, w_2, \cdots, w_N]$.

EXAMPLE 5.28: Here are a few simple examples of weighted voting systems:
(a) [7: 5, 2, 1]. In this system, a motion wins if, and only if V_1 and V_2 vote for it. V_3's vote is thus irrelevant; such a voter is called a **dummy voter**.
(b) [9: 5, 5, 4, 2, 1]. In this system, in order for a motion to pass, either V_1 or V_2 must vote for it.
(c) [10: 8, 3, 2, 2, 1]. In this system, voter V_1 has **veto power** since in order for any motion to pass it must have V_1's vote.

The above example makes it clear that some serious inequities might arise in weighted voting systems. The examples were small enough so that this was evident, but when the number of voters exceeds even a moderate size such as 30, the analysis of weighted voting systems can become extremely complex. Different notions of assigning a certain percentage of the "power" to the voters have been developed. One of the more widely accepted methods was developed by John Banzhaf III,[10] and is described in the following definition.

[10] John Banzhaf III (1940–) is a law professor at George Washington University. The "Banzhaf index" was actually invented by Lionel Penrose in a 1946 statistics paper, but went largely unnoticed by the scientific community. Banzhaf rediscovered and popularized the index in a seminal 1965 paper

DEFINITION 5.8: (*Banzhaf Power Index*) Suppose that we are given a weighted voting system $[q: w_1, w_2, \cdots, w_N]$. A **coalition** is a nonempty set S of voters (who may vote together) and its **weight** $\mathrm{wgt}(S)$ is simply the sum of the weights of its individual voters. A coalition is a **winning coalition** if its weights add up to at least q, otherwise it is a **losing coalition**. A voter in a winning coalition is **critical** (for that coalition) if without him/her the coalition would become a losing one. For each voter V_n, we define $c(V_n)$ to be the total number of winning coalitions in which V_n is critical, and we let $T = \sum_{n=1}^{N} c(V_n)$. The **Banzhaf index** of a voter V_n is defined to be $c(V_n)/T$.

EXAMPLE 5.29: Compute the Banzhaf indices of each voter in the weighted voting system [7: 5, 2, 1] of Example 5.28(a).

SOLUTION: We analyze all coalitions for the given weighted voting system:

Coalition	Weight	Winning?	Critical Voters
$\{V_1\}$	5	No	N/A
$\{V_2\}$	2	No	N/A
$\{V_3\}$	1	No	N/A
$\{V_1, V_2\}$	7	Yes	V_1, V_2
$\{V_1, V_3\}$	6	No	N/A
$\{V_2, V_3\}$	3	No	N/A
$\{V_1, V_2, V_3\}$	8	Yes	V_1, V_2

Thus we have $c(V_1) = c(V_2) = 1,\ c(V_3) = 0,\ T = 2,$ so the Banzhaf indices of V_1, V_2 are both 1/2, and that of V_3 is zero. This can be interpreted by saying that for all practical purposes, V_1, V_2 share an equal amount of power in this weighted voting system, despite the fact that V_1 controls 150% more votes than V_2.

EXERCISE FOR THE READER 5.25: Compute the Banzhaf indices of each voter in the weighted voting system [9: 5, 5, 4, 2, 1] of Example 5.28(b).

The following exercise for the reader asks to compute the Banzhaf indices for the historically significant setting of the 1964 Nassau County voting system that motivated Banzhaf to develop his theory.

EXERCISE FOR THE READER 5.26: (a) Compute the Banzhaf indices of each voter in the Nassau County NY Board of Supervisors weighted voting system [59: 31, 31, 28, 21, 2, 2], then (b) identify any voters who are *dictators* (any voter whose weight is greater to or equal to the quota), dummies, or have veto power.

The brute-force approach used in Example 5.29 quickly becomes impractical since it generally requires checking through $2^N - 1$ coalitions. For example, to compute the Banzhaf indices for the 50 states and the District of Columbia with respect to the electoral votes accorded to each state (which are computed using the latest census figures), this approach would require looking at over 2 quadrillion

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entitled "Why weighted voting doesn't work," in which he demonstrated that a certain voting system in Nassau County, NY gave "power" to only three out of the six voting districts.

coalitions, which, at the time of the writing of this book, would be intractable to perform in a reasonable amount of time even on a supercomputer.[11]

Generating functions can be used to render a much more efficient algorithm for computing Banzhaf indices. The method is a bit more involved than the previous generating function counting schemes that we have introduced; the details of its development will be the subject of the following example:

EXAMPLE 5.30: Develop a method involving generating functions to compute the Banzhaf indices of each voter in a weighted voting system $[q : w_1, w_2, \cdots, w_N]$.

SOLUTION: In case any of the weights w_n is at least equal to the quota q, then the corresponding voter is a dictator since his/her vote alone decides the overall decision. Since such weighed voting systems are not very interesting, we henceforth make the following:

Assumption: $w_n < q$ for each n $(1 \le n \le N)$.

We begin with the following generating function:

$$F(x) = (1 + x^{w_1})(1 + x^{w_2}) \cdots (1 + x^{w_N}). \tag{13}$$

We observe from (13) that since the term of highest degree in the expanded form of $F(x)$ is $x^{w_1 + w_2 + \cdots + w_N}$, it follows that the degree of $F(x)$ is $W \triangleq \sum_{n=1}^{N} w_i$. (The total weight of all votes.) We introduce notation for the coefficients of the expanded form of the generating function by means of the following equation:

$$F(x) = a_W x^W + a_{W-1} x^{W-1} + \cdots + a_2 x^2 + a_1 x + a_0. \tag{14}$$

We have already observed that $a_W = 1$; similarly (13) shows that $a_0 = 1$. The following observation is seen by comparing (13) and (14) and interprets the coefficients of the generating function in terms of the voting system:

Key Observation: a_i = the number of coalitions of weight i.

We first will develop a recursive formula for these important generating function coefficients. The idea will be to build up the generating function $F(x)$ through a sequence of N factor multiplications. More precisely, we define the following sequence of generating functions:

$$\begin{aligned}
&F_0(x) = 1 \\
&F_1(x) = (1 + x^{w_1}) \\
&F_2(x) = (1 + x^{w_1})(1 + x^{w_2}) \\
&F_3(x) = (1 + x^{w_1})(1 + x^{w_2})(1 + x^{w_3}) \\
&\qquad \vdots \\
&F_N(x) = (1 + x^{w_1})(1 + x^{w_2}) \cdots (1 + x^{w_N}) = F(x).
\end{aligned}$$

[11] On his Web site, Banzhaf has a link to some data sets including computations of the 51 Banzhaf indices for this electoral college system. In contrast to the efficient exact method that we will introduce, Banzhaf employed an approximation method that used simulations to randomly generate 4.29 billion coalitions, and it took nearly 25 hours on his computer. Simulations will be explained in Section 6.2.

We note that

$$F_j(x) = F_{j-1}(x)(1 + x^{w_j}), \text{ for } 1 \le j \le N. \tag{15}$$

For each index j, $0 \le j \le N$, we define integers $a_i^{(j)}, (0 \le i \le W)$ to be the coefficients of the expansion of $F_j(x)$, and thus we may write:

$$F_j(x) = a_W^{(j)} x^W + a_{W-1}^{(j)} x^{W-1} + \cdots + a_2^{(j)} x^2 + a_1^{(j)} x + a_0^{(j)}. \tag{16}$$

If we define $a_i^{(j)} = 0$ if *i* is a negative integer, it follows by comparing coefficients of x^i in both sides of (15) that the following recursion formula is valid:

$$a_i^{(j)} = a_i^{(j-1)} + a_{i-w_j}^{(j-1)} \quad (0 \le i \le W, 1 \le j \le N). \tag{17}$$

We have thus developed an effective recursion scheme for computing the coefficients $a_i^{(j)}$ (and hence also the coefficients $a_i = a_i^{(N)}$), but how is this going to help us to compute the desired Banzhaf indices? We next show how these coefficients can help us to compute another set of coefficients from which we <u>will</u> be able to directly obtain the Banzhaf indices; this latter process will be the novelty of the method. To this end, we temporarily focus attention on a certain voter $V_n\left(1 \le n \le N\right)$. It will be convenient to introduce the following notation:

NOTATION: We let σ_n be the number of **swings** for voter V_n; in other words, this is the number of losing coalitions *S* such that $V_n \notin S$, that would become winning coalitions if V_n were to join.

Since $\mathrm{wgt}(V_n) = w_n$, it follows that[12]

$$\sigma_n = |\{S \subseteq \{V_1, V_2, \cdots, V_N\} : V_n \notin S \text{ and } q - w_n \le \mathrm{wgt}(S) < q\}|. \tag{18}$$

Notice that $\sigma_n = c(V_n)$. Each of these coefficients is the sum of the following coefficients:

$$b_i = |\{S \subseteq \{V_1, V_2, \cdots, V_N\} : V_n \notin S \ \mathrm{wgt}(S) = i\}|, \tag{19}$$

that is

$$\sigma_n = \sum_{i=q-w_n}^{q-1} b_i. \tag{20}$$

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[12] Although, by definition, coalitions are required to be nonempty, the condition $S \ne \varnothing$ would be a redundancy in (18) since it is already required that $q - w_n < \mathrm{wgt}(S)$ and we are assuming throughout the development that $w_n < q$ (thus the latter condition implies that $\mathrm{wgt}(S) > 0$, so *S* cannot be empty).

From the σ_n 's the Banzhaf indices are easily computed:

$$\text{Banzhaf index of voter } V_n = \frac{\sigma_n}{\sigma_1 + \sigma_2 + \cdots + \sigma_N}. \tag{21}$$

We could compute the b_i 's in the same fashion that was shown for the a_i 's (using a corresponding generating function with one less weight), but there is a more efficient scheme. If we define $W_n = W - w_n$ (i.e., the total voting weight less the weight of voter V_n), then reordering and taking V_n last, it follows that we may write:

$$\begin{aligned} F(x) &= a_W x^W + a_{W-1} x^{W-1} + \cdots + a_2 x^2 + a_1 x + a_0 \\ &= (b_{W_n} x^{W_n} + b_{W_n - 1} x^{W_n - 1} + \cdots + b_2 x^2 + b_1 x + b_0)(1 + x^{w_n}) \end{aligned} \tag{22}$$

(This really just follows from the recursion formula (15), if we reorder the voters so that V_n is taken last.) If, as was already done for the a_i 's, we define $b_i = 0$ whenever i is a negative integer, then by comparing coefficients of x^i in both sides of (22) we obtain the following recursion formula:

$$b_i = a_i - b_{i-w_n} \ (i = 0, 1, 2, \ldots). \tag{23}$$

By appropriately combining the preceding recursion formulas, we arrive at the following algorithm for the computation of Banzhaf indices.

ALGORITHM 5.1: (*Generating Function Based Recursion Algorithm for Computing Banzhaf Indices in a Weighted Voting System*)
Input: A weighted voting system $[q : w_1, w_2, \cdots, w_N]$, where the weight w_n of voter V_n is less than the quota q (so there are no dictators).
Output: The corresponding Banzhaf indices of the N voters.

Step 1: (*Initialize known coefficients*)
Set $a_0 = a_W = 1$. Set $a_0^{(0)} = 1$ and $a_i^{(0)} = 0$, for each $i \neq 0$.
Set $a_i^{(j)} = b_i = 0$, whenever index i is negative.

Step 2: (*Compute the $a_i^{(j)}$'s*)
FOR index j = 1 TO N
    FOR index $i = 0$ TO $i = W - 1$
        Set $a_i^{(j)} = a_i^{(j-1)} + a_{i-w_j}^{(j-1)}$ (using (17))
    END i FOR
END j FOR

Step 3: (*Record the a_i 's*)
FOR index $i = 1$ TO $i = W - 1$
    Set $a_i = a_i^{(N)}$
END i FOR

Step 4: (*Compute the* σ_n *'s*)
FOR index n = 1 TO N
(*First find the needed* b_i *'s corresponding to voter* V_n)
FOR index i = 0 TO $q-1$
Set $b_i = a_i - b_{i-w_n}$
END i FOR
(*Record* σ_n)
$\sigma_n = \sum_{i=q-w_n}^{q-1} b_i$
END n FOR

Step 5: (*Compute the Banzhaf indices*)
Set $T = \sum_{n=1}^{N} \sigma_n$
FOR index n = 1 TO N
Banzhaf index of $V_n = \sigma_n / T$
END n FOR

Of course, for small sized voting systems, the overhead of this algorithm would make it more cumbersome than the brute-force method. But notice that the computational steps 2, 4, and 5 respectively take at most NW, Nq, and $2N-1$ mathematical operations (additions, subtractions, divisions), which total **less than** $3NW$ operations. Recall that N is the number of voters, and W is the total weight of the votes. Compare this with the amount of work needed in the brute-force approach that was used in Example 5.29. In general, all but the nonempty set of voters needs to be considered, and there are $2^N - 1$ of these. Also, for each coalition, a nontrivial amount of work needs to get done (check its weight and determine whether it is winning, and if it is, determine the critical voters). So this method requires **more than** 2^N mathematical operations. In the electoral college example that Banzhaf considers on his Web site, N = 51 (50 states and the District of Columbia), and W = 538 (total number of electoral votes). Thus, the brute-force approach would require (much) more than $2^{51} = 2.25... \times 10^{15}$ mathematical operations, whereas Algorithm 5.1 would require at most $3 \cdot 51 \cdot 538 = 82{,}314$ mathematical operations. Thus, although as Banzhaf had found, the brute-force approach is impossible to do in a reasonable amount of time (even with a computer), Algorithm 5.1 would be well suited for this example. The computer implementation material at the end of this chapter will consider some specific applications of Algorithm 5.1.

The following exercise for the reader asks to compute the Banzhaf indices both directly (by considering all coalitions) and by using the generating function method for the historically significant setting of the 1964 Nassau County voting system that motivated Banzhaf to develop his theory. The size of the example is small enough to do by hand, and for the speed of the generating function method to not yet be realized. Some larger examples that are feasible only with the latter method will be considered in the computer exercises and implementation material that follows this appendix.

EXERCISE FOR THE READER 5.27: Use Algorithm 5.1 to recompute the Banzhaf indices of each voter in the weighted voting system [7: 5, 2, 1] of Example 5.28(a).

In analyzing voting systems, particularly with generating function methods, it often occurs that the weights of the votes in the system can be reduced while preserving the essential features of the system. As a very simple example, everyone should immediately agree that the voting system [4: 2, 2, 2, 2] is equivalent to [2: 1, 1, 1, 1], in that all possible voting scenarios in either system would have the same result in either system. The following definition generalizes this concept.

DEFINITION 5.9: (*Equivalent Voting Systems*) Two weighted voting systems $[q : w_1, w_2, \cdots, w_N]$ and $[q' : w_1', w_2', \cdots, w_{N'}']$ are said to be equivalent if the number of voters is the same, and any coalition of voters in the first is winning if, and only if it is winning in the second.

Since the running time of Algorithm 5.1 depends on the total weight W of all votes, when applying this algorithm, it is desirable to find a voting system equivalent to the one being analyzed and with as small a total weight as possible.

EXERCISE FOR THE READER 5.28: In 1994, the Nassau County NY Board of Supervisors modified their weighted voting system to the following: [65: 30, 28, 22, 17, 7, 6].
(a) Show that [15:7, 6, 5, 4, 2, 1] is an equivalent voting system.
(b) Apply the generating function method to compute the Banzhaf power indices using the original system, and then the equivalent system of Part (a).
(c) Show that there does not exist an equivalent voting system of smaller weight than that given in Part (a). Thus, the system given in Part (a) is called a *minimum equivalent voting system* to the given system.

Although it is desirable to have equivalent voting systems of reduced total weight, there does not seem to exist an efficient algorithm for their determination.

ADDITIONAL EXERCISES FOR THE APPENDIX TO SECTION 5.3:

1. For each voting system given, do the following. Compute all Banzhaf power indices by: (i) Analyzing each coalition for its critical voters, as in the solution of Example 5.29. (ii) Using generating functions. (iii) Then identify any voters who are dictators, dummies, or have veto power.
   (a) [4: 2, 1, 1, 1, 1] (b) [12: 8, 5, 5, 3, 2]

2. For each voting system given, do the following. Compute all Banzhaf power indices by: (i) Analyzing each coalition for its critical voters, as in the solution of Example 5.29. (ii) Using generating functions. (iii) Then identify any voters who are dictators, dummies, or have veto power.
   (a) [6: 4, 2, 2, 1, 1] (b) [22: 14, 12, 10, 8, 5]

3. For each statement below regarding voting systems, either explain why it is (always) true, or provide an example of a weighted voting system in which it is false.
   (a) If the Banzhaf power index of a voter is greater than 1/2, then the voter must be a dictator.
   (b) If there are N voters, then any voter with veto power must have Banzhaf power index at least $1/N$.

COMPUTER IMPLEMENTATIONS AND EXERCISES FOR SECTION 5.3

(***Polynomial Arithmetic on Computers***) Readers who are using computing platforms with so-called symbolic or computer algebra capabilities (such as MAPLE, Mathematica®, and MATLAB®) will be able to perform the polynomial multiplications and additions/subtractions needed to compute coefficients of generating functions. But it is not hard to use any computing system to perform this sort of arithmetic. We will show how to represent polynomials as vectors and some corresponding efficient schemes for performing arithmetic on them. In all of our applications, the polynomials will have integer coefficients, so this may be assumed in what follows.

It is often convenient to store a polynomial $f = a_n x^n + a_{n-1}x^{n-1} + \cdots a_1 x + a_0$ as the vector of its coefficients: $f = \sum_{i=0}^{n} a_i x^i \sim [a_n, a_{n-1}, \cdots, a_1, a_0]$. This basic idea is so important that we repeat it with emphasis:

Polynomial	Vector of Coefficients
$a_n x^n + a_{n-1}x^{n-1} + \cdots a_1 x + a_0$	$[a_n, a_{n-1}, \cdots, a_1, a_0]$

The addition and multiplication operations can be converted into corresponding operations on such vectors. Suppose that $g = \sum_{i=0}^{m} b_i x^i \sim [b_m, b_{m-1}, \cdots, b_1, b_0]$ is another polynomial. From the definition of addition of polynomials, we may write:

$$f + g \sim [c_N, c_{N-1}, \cdots, c_1, c_0], \text{ where } N = \max(n, m) \text{ and } c_i = a_i + b_i, \tag{24}$$

for $1 \le i \le N$. (We adhere to the convention made in the section that unspecified coefficients are zero.)

To understand the vector version of polynomial multiplication, we first see how it will work if we multiply $f \neq 0$ by a **monomial,** which is a nonzero polynomial consisting of a single term: $b_k x^k$

$$f \cdot b_k x^k = (a_n x^n + a_{n-1}x^{n-1} + \cdots a_1 x + a_0) \cdot x^k = b_k [a_n x^{n+k} + a_{n-1}x^{n+k-1} + \cdots a_1 x^{1+k} + a_0 x^k]$$

In vector notation, this multiplication becomes:

$$[a_n, a_{n-1}, \cdots, a_1, a_0] \cdot [b_k, \underbrace{0, 0, \cdots, 0}_{k \text{ zeros}}] = b_k [a_n, a_{n-1}, \cdots, a_1, a_0, \underbrace{0, 0, \cdots, 0}_{k \text{ zeros}}]. \tag{25}$$

By (repeatedly) using the distributive law, a general polynomial multiplication can be broken down into a sum of multiplications of a polynomial by a monomial:

$$f \cdot g = f \cdot \sum_{i=0}^{m} b_i x^i = \sum_{i=0}^{m} f \cdot b_i x^i \Rightarrow f \cdot g \sim \sum_{i=0}^{m} b_i [a_n, a_{n-1}, \cdots, a_1, a_0, \underbrace{0, 0, \cdots, 0}_{i \text{ zeros}}]. \tag{26}$$

Thus, with this method of storing polynomials along with the associated algorithms (24) and (26) for their addition and multiplication, we have an efficient means for manipulating polynomials on computing platforms. This will serve as a basis for the computer implementation such polynomial arithmetic and thus in their use in computing coefficients of generating functions.

1. (*Program for Polynomial Addition*) (a) Write a program with syntax:

   `Sum = PolyAdd(px,qx)`

   that will add two polynomials. The two inputs, `px` and `qx` are vectors representing the polynomials to be added. The output, `Sum`, is a vector representing the sum of the inputted polynomials. If the sum is the zero polynomial, the output should be [0]; otherwise, the output should have a nonzero first component (so that the degree of the sum is one less than the length of the output vector).

   (b) Run your program on the following polynomial additions:

   (i) $(x^5 + x + 1) + (x^8 + x^6 + 4x^2 + 2)$

   (ii) $(x^3 + 2x^2 + 1) + (x^8 - x^7 + x^6 - x^5 + x^4 - x^3 + x^2 - x + 1)$

2. (*Program for Polynomial Multiplication*) (a) Write a program with syntax:

   `Prod =PolyMult(px,qx)`

   that will multiply two polynomials. The two inputs, `px` and `qx` are vectors representing the polynomials to be multiplied. The output, `Prod`, is a vector representing the product of the

inputted polynomials. If the product is the zero polynomial, the output should be [0]; otherwise, the output should have a nonzero first component (so that the degree of the sum is one less than the length of the output vector).
(b) Run your program on the following polynomial multiplications:
(i) $(x^5 + x + 1) \cdot (x^8 + x^6 + 4x^2 + 2)$
(ii) $(x^3 + 2x^2 + 1) \cdot (x^8 - x^7 + x^6 - x^5 + x^4 - x^3 + x^2 - x + 1)$

3. Making use of the programs of either of the preceding two computer exercises, compute the coefficient of x^{10} in each of the following generating functions:
(i) $(x^8 - x^7 + x^6 - x^5 + x^4 - x^3 + x^2 - x + 1)^4$
(ii) $1/[(1-x)(1-x^2)(1-x^3)]$

4. (a) Determine the number of ways that $1000 could be distributed using bills of any or all of the following denominations: $1, $5, $10, $20, $50, $100.
(b) Suppose that we roll 10 regular dice, and we add up all of the numbers that show (so the number will lie between 6 and 60, inclusive). How many different ways could a total of 15 show up? How about a total of 30?

5. (*Program for Partition Function Based on Generating Functions*) (a) Write a program with syntax:

```
pn = Partition(n)
```

that will input, `n`, a positive integer, and will output, `pn`, the number of partitions of n, i.e., $p(n)$. Use the generating function of Part (a) of Ordinary Exercise 27 with $m = n$, the expansion (8) (multiple times with powers of x replacing x), and repeatedly use the program of Computer Exercise 2.
(b) Use your function from Part (a) to compute the terms of the sequence $p(5), p(10), p(15), p(20), \cdots$ until it takes a term more than 1 minute to execute.
(c) Create a more efficient program than that of Part (a):

```
pn = PartitionVer2(n)
```

by modifying the program of Computer Exercise 2 so that coefficients of powers higher than x^n are ignored.
(d) Repeat Part (b) using instead the program of Part (c).

6. (*Program for Partition Function Not Based on Generating Functions*) (a) Write a program with syntax:

```
pn = PartitionBrute(n)
```

having the same input/output as that of Part (a) of the preceding computer exercise, but in this exercise do not base the program on the generating function approach. Use either a brute-force approach, or whatever other method you can think of.
(b) Use your function of Part (a) to compute the terms of the sequence $p(5), p(10), p(15), p(20), \cdots$ until a term takes more than 1 minute to execute. Compare with the performance of the generating function based program.

7. (*Program for Odd Partition Function Based on Generating Functions*) (a) Write a program with syntax:

```
pOddn = OddPartition(n)
```

that will input, `n`, a positive integer, and will output, `pOddn`, the number of odd partitions of n, i.e., the function $p_O(n)$ introduced in ordinary Exercise 28(b). Use the generating function that was found in Exercise 28(b).

(b) Use your function of Part (a) to compute the terms of the sequence $p_O(5), p_O(10), p_O(15), p_O(20), \cdots$ until a term takes more than 1 minute to execute.

(c) Create a more efficient program than that of Part (a):

```
pOddn = OddPartitionVer2(n)
```

by modifying the program of Computer Exercise 2 so that coefficients of powers higher than x^n are ignored.

(d) Repeat Part (b) using instead the program of Part (c).

NOTE: The remaining computer exercises deal with the material from the appendix to Section 5.3.

8. (*Program for Computation of Banzhaf Power Indices: Brute-force Version*) (a) Write a program with syntax:

```
BIndVec = BanzhafBrute(q,WVec)
```

whose two input variables correspond to the parameters of a weighted voting system: `q`, a positive integer representing the quota, and `WVec`, a vector of positive integers representing the voter weights. The output, `BIndVec`, will be the corresponding vector of Banzhaf power indices. The program should follow the brute-force method that was used in Example 5.29, i.e., by considering all coalitions, identifying critical voters for each, and keeping separate tallies for each voter.

(b) Use your function from Part (a) to compute the Banzhaf power indices for the weighted voting system of Example 5.29 and for the Nassau County Board of Supervisors system of Exercise for the Reader 5.26.

(c) For each positive even integer *n*, we define a weighted voting system $\mathscr{V}_n$ defined by:

$$\mathscr{V}_n = [n(n-1)/2;\, n, n-2, n-4, \cdots, 2]$$

Repeatedly apply your program of Part (a) to the systems $\mathscr{V}_n$ with n = 10, 12, 14, … until the program takes longer than one minute to execute.

9. (*Program for Computation of Banzhaf Power Indices: Generating Function Version*) (a) Write a program with syntax:

```
BIndVec = BanzhafGF(q,WVec)
```

whose input/output variables are the same as in the preceding computer exercise, but that implements Algorithm 5.1.

(b) Use your function from Part (a) to compute the Banzhaf power indices for the weighted voting system of Example 5.29 and for the Nassau County Board of Supervisors system of Exercise for the Reader 5.26.

(c) For each positive even integer *n*, we define a weighted voting system $\mathscr{V}_n$ defined by:

$$\mathscr{V}_n = [n(n-1)/2;\, n, n-2, n-4, \cdots, 2]$$

Repeatedly apply your program of Part (a) to the systems $\mathscr{V}_n$ with n = 10, 12, 14, … until the program takes longer than one minute to execute. Compare with the results of Part (c) of the preceding computer exercise.

10. (*Computation of Banzhaf Power Indices in Some Well-Known Voting Systems*) In this exercise, you are to apply your program from Computer Exercise 9 to compute the Banzhaf power indices of the well known weighted voting systems that are described below. You will need to obtain or download the relevant data from the internet.

(a) The European Community weighted voting system that was established by the Treaty of Nice that went into effect in 2004.

(b) The 51 voters (50 states and the District of Columbia) of the United States Electoral College.

≡

CHAPTER 5: COUNTING TECHNIQUES, COMBINATORICS, AND GENERATING FUNCTIONS

EFR 5.1: Part (a): $12\cdot 11\cdot 10\cdot 9 \quad \cdot 8 = 95{,}040$ Part (b): With neither K nor S, there are (as in Part (a) but with the remaining 10 players to choose from) $10\cdot 9\cdot 8\cdot 7\cdot 6 = 30{,}240$ possible lineups. Now for the (disjoint) case in which exactly one of K or S appears, we use the multiplication principle as follows:

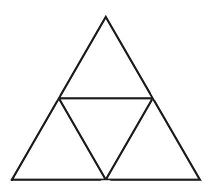
$$\underset{\substack{\text{choices for}\\ \text{K or S}}}{2} \cdot \underset{\substack{\text{choices for}\\ \text{where to put}\\ \text{K or S}}}{5} \cdot \underset{\substack{\text{ways to fill the}\\ \text{remaining four slots}\\ \text{with remaining 10 players}}}{10\cdot 9\cdot 8\cdot 7} = 50{,}400.$$

This gives a total of 80,640 lineups.

EFR 5.2: Invoking the notation of the solution of Example 5.6, we are looking for $|D_2 \cup D_3 \cup D_5 \cup D_{11}|$. Using the inclusion-exclusion principle along with the facts pointed out in the solution of Example 5.6 that $D_n \cap D_m = D_{\text{lcm}(n,m)}$ (and its easy extension to larger intersections by induction), and $|D_n| = \lfloor 3600/n \rfloor$, we obtain:

$$\begin{aligned}
|D_2 \cup D_3 \cup D_5 \cup D_{11}| &= |D_2|+|D_3|+|D_5|+|D_{11}| - \big[|D_6|+|D_{10}|+|D_{22}|+|D_{15}|+|D_{33}|+|D_{55}|\big] \\
&= +\big[|D_{30}|+|D_{66}|+|D_{110}|+|D_{165}|\big] - |D_{330}| \\
&= \lfloor 3600/2 \rfloor + \lfloor 3600/3 \rfloor + \lfloor 3600/5 \rfloor + \lfloor 3600/11 \rfloor - \big[\lfloor 3600/6 \rfloor + \lfloor 3600/10 \rfloor \\
&= + \lfloor 3600/22 \rfloor + \lfloor 3600/15 \rfloor + \lfloor 3600/33 \rfloor + \lfloor 3600/55 \rfloor\big] + \big[\lfloor 3600/30 \rfloor + \\
&= + \lfloor 3600/66 \rfloor + \lfloor 3600/110 \rfloor + \lfloor 3600/165 \rfloor\big] - \lfloor 3600/330 \rfloor \\
&= 1800+1200+720+327-[600+360+163+240+109+65] \\
&= +[120+54+32+21]-10 \\
&= 2727.
\end{aligned}$$

EFR 5.3: Each of the three strings of letters can be made into $10^3 = 1000$ license plates, and these are all different. Thus by the complement principle, we need only subtract the total number (3000) of these plates from the total number of Hawaii plates that was found in Example 5.2 $(17{,}576{,}000)$ to get the answer to this question: $17{,}573{,}000$.



EFR 5.4: Part (a): We subdivide the equilateral triangle into four smaller ones with side length 1/2 (see figure). Since there are five points in the larger triangle, at least two must lie in a single smaller triangle. Since the diameter

of any triangle is its longest side length, it follows that two such points will lie at a distance of at most 1/2 from each other.
Part (b): The only way for two points on a triangle to have distance between them equaling the length of the (longest) side length is for the two points to be endpoints of such a side. Thus, for the two points of Part (a) to have separation distance equal to 1/2, they would have to lie at the endpoints of a side of one of the four smaller triangles, and this would put them on an edge of the original (larger) triangle.

EFR 5.5: Let $a_1, a_2, \cdots, a_{51}$ be the 51 positive integers. For each a_i, we write $a_i = 2^{p_i} b_i$, where b_i is an odd integer. Since there are exactly 50 odd integers between 1 and 100, it follows from the pigeonhole principle that at least two of the a_i 's must share the same odd integer: i.e., $b_i = b_j$, for two indices $1 \le i \ne j \le 51$. It follows that either $a_i \mid a_j$ (if $p_i \le p_j$) or $a_j \mid a_i$. □

EFR 5.6: By Proposition 2.2, any integer must fall into one of *n* equivalence classes in the equivalence relation of congruence modulo *n*. Therefore by the pigeonhole principle, in a set of *n* + 1 integers, there must be two, *a* and *b*, that lie in the same equivalence class (mod *n*). Thus $a \equiv b \pmod{n}$, and this means (by definition) that *n* divides $a - b$.

EFR 5.7: Part (a): Since arrangements that are obtainable from one another by rotations are considered equivalent, in counting all arrangements, we can place one particular person, call him/her X in some particular seat (since in any arrangement, X could always be brought to this seat with a rotation), and proceed, say clockwise, to fill the remaining $n-1$ seats with the remaining $n-1$ people. There are $(n-1)!$ such permutations of the remaining people, and all give nonequivalent seating arrangements of the whole group.
Part (b): We use the idea of Part (a). Take X as a man, then the multiplication principle tells us that to fill the remaining $n-1 = 2k-1$ seats with alternating men and women, there are $k \cdot (k-1) \cdot (k-1) \cdot (k-2) \cdot (k-2) \cdot \cdots \cdot 2 \cdot 2 \cdot 1 \cdot 1 = k! \cdot (k-1)!$ ways to do this.
Part (c): Put Jimmy in a particular seat. Now there are two choices where to put Sue, either counterclockwise or clockwise next to Jimmy. After this choice has been made, of the $n-2$ remaining seats, the multiplication principle tells us there will be $(k-1) \cdot (k-1) \cdot (k-2) \cdot (k-2) \cdot \cdots \cdot 2 \cdot 2 \cdot 1 \cdot 1 = [(k-1)!]^2$ ways to fill these. Thus (again by the multiplication principle), there are $2 \cdot [(k-1)!]^2$ different seating arrangements where Jimmy and Sue are sitting next to each other.
Part (d): Put Jimmy in a particular seat. Next, fill the counterclockwise and clockwise seats next to Jimmy; by the multiplication principle, there are $(k-1) \cdot (k-2)$ ways to do this with women other than Sue. Now, the remaining $n-3$ seats (clockwise) can be filled in any way (alternating the $k-1$ men and $k-2$ women), so there are a total of $(k-1) \cdot (k-2) \cdot (k-2) \cdot (k-3) \cdot \cdots \cdot 2 \cdot 1 \cdot 1 = (k-1)! \cdot (k-2)!$ ways to do this. Combining these counts (again with the multiplication principle) tells us that there are a total of $(k-2) \cdot [(k-1)!]^2$ different arrangements.
NOTE: The reader should observe that the answers in Parts (c) and (d) should add up to the answer in Part (b), and verify that this is the case.

EFR 5.8: Part (a): The multiplication principle allows us to count the number of full houses as follows:

$$\underbrace{13}_{\substack{\text{Number of ways}\\ \text{to choose denomination}\\ \text{of the three of a kind}}} \cdot \underbrace{12}_{\substack{\text{Number of ways}\\ \text{to choose denomination}\\ \text{of the two of a kind}}} \cdot \underbrace{C(4,3)}_{\substack{\text{Number of ways}\\ \text{to choose 3 cards}\\ \text{from first denom.}}} \cdot \underbrace{C(4,2)}_{\substack{\text{Number of ways}\\ \text{to choose 2 cards}\\ \text{from second denom.}}} = 13 \cdot 12 \cdot 4 \cdot 6 = 3744.$$

Part (b): As in Part (a), the multiplication principle yields the number of flushes to be

$$\underbrace{4}_{\substack{\text{Number of ways}\\\text{to choose the suit}\\\text{of the flush}}} \cdot \underbrace{C(13,5)}_{\substack{\text{Number of ways}\\\text{to choose 5 cards}\\\text{from this suit}}} \cdot = 4\cdot 1287 = 5148.$$

Part (c): Once again, the multiplication principle can be used to tally the number of four of a kind poker hands

$$\underbrace{13}_{\substack{\text{Number of ways}\\\text{to choose denomination}\\\text{of the four of a kind}}} \cdot \underbrace{C(4,4)}_{\substack{\text{The number of ways}\\\text{to choose (all) four cards}\\\text{from this denom.} = 1.}} \cdot \underbrace{48}_{\substack{\text{Number of cards}\\\text{to choose}\\\text{to complete the hand}}} = 624.$$

<u>EFR 5.9</u>: We use the identity (4) to translate binomial coefficient into factorial expressions, manipulate these and then translate back:

$$\binom{n-1}{k-1}+\binom{n-1}{k}=\frac{(n-1)!}{(k-1)!(n-1-[k-1])!}+\frac{(n-1)!}{(k)!(n-1-k)!} = \frac{(n-1)!}{(k-1)!(n-k)!}+\frac{(n-1)!}{(k)!(n-1-k)!}$$

$$=\frac{k\cdot(n-1)!}{k\cdot(k-1)!(n-k)!}+\frac{(n-1)!(n-k)}{(k)!(n-1-k)!(n-k)} = \frac{[k+n-k]\cdot(n-1)!}{k!(n-k)!}=\frac{n\cdot(n-1)!}{k!(n-k)!}$$

$$=\frac{n!}{k!(n-k)!}=\binom{n}{k}.$$

<u>EFR 5.10</u>: Part (a): If we apply the binomial theorem to expand $0=(-1+1)^m$, i.e., use $x=-1$ and $y=1$ in (6), we obtain

$$0=(-1+1)^m=\sum_{k=0}^{m}\binom{m}{k}(-1)^k(1)^{m-k}=\sum_{k=0}^{m}\binom{m}{k}(-1)^k=\binom{m}{0}-\binom{m}{1}+\binom{m}{2}-\cdots\pm\binom{m}{m}.$$

Since $\binom{m}{0}=1$, subtracting the other terms from both sides of the above equation produces the desired identity.

Part (b): Consider $x\in\bigcup_{i=1}^{n}A_i$, and suppose that x lies in exactly m of the A_i 's. Then the counts for x from all of the terms of the right-hand side of (1) will only go up to $a = m$ (corresponding to intersections of m A_i 's). The contribution of x from the summation in the ath term (with $1\le a\le m$)

$$(-1)^{a+1}\sum_{i_1<i_2<\cdots<i_a}|A_{i_1}\cap A_{i_2}\cap\cdots\cap A_{i_a}|$$

will equal to the number of ways of choosing the a indices $i_1<i_2<\cdots<i_a$ from $\{1,2,\cdots,n\}$ so that they all correspond to sets A_i which contain x. Since there are m such A_i 's, it follows that the contribution of x from this term must be $(-1)^{a+1}\binom{m}{a}$. Adding up all of these contributions (from $a = 1$ to $a = m$) gives exactly the sum $\binom{m}{1}-\binom{m}{2}+\cdots\pm\binom{m}{m}$, which, by Part (a) simply equals 1. Since this is true for any x in $\bigcup_{i=1}^{n}A_i$, while for any x outside $\bigcup_{i=1}^{n}A_i$, the contribution of x on the right (and the left) is clearly zero, the identity is established.

EFR 5.11: Of the 11 letters in the word MISSISSIPPI, only four are different: there is one M, four I's, four Ss, and two Ps. Thus, by Theorem 2.8, the number of distinguishable permutations of these 11 letters is $\binom{11}{1,4,4,2} = \frac{11!}{1!4!4!2!} = \frac{\cancel{4!}5\cdot\cancel{6}\cdot 7\cdot\cancel{8}\cdot 9\cdot 10\cdot 11}{\cancel{4!}\,\cancel{2\cdot 3}\cdot\cancel{4\cdot 2}} = 34{,}650.$

EFR 5.12: We give a combinatorial proof similar to what was done in our proof of the binomial theorem (Theorem 5.7). If we expand the left-hand side of (8):

$$(x_1 + x_2 + \cdots + x_r)^n = \underbrace{(x_1 + x_2 + \cdots + x_r)\cdot(x_1 + x_2 + \cdots + x_r)\cdots\cdot(x_1 + x_2 + \cdots + x_r)}_{n \text{ factors}},$$

the resulting expansion will consist of all terms that are products of the form $z_1 z_2 \cdots z_n$, where each z_i is a single term selected from the ith factor above (so each z_i must be one of $x_1, x_2, \cdots, x_r$). Thus, each term in the expansion will be of the form $x_1^{k_1} x_2^{k_2} \cdots x_r^{k_r}$, where the exponents are nonnegative integers that add up to n. By Theorem 5.8, the number of occurrences of this term in the expansion will equal $\binom{n}{k_1, k_2, \cdots, k_r}$. Putting these facts together shows that the expansion equals the right-hand side of (8), as desired. □

EFR 5.13: In the multinomial identity (8):

$$(x_1 + x_2 + \cdots + x_r)^n = \sum_{\substack{k_1 + k_2 + \cdots + k_r = n \\ k_i \text{ nonnegative integer}}} \binom{n}{k_1, k_2, \cdots, k_r} x_1^{k_1} x_2^{k_2} \cdots x_r^{k_r},$$

if we specialize to the case that $r = 2$, and put $x_1 = x$ and $x_2 = y$, and use the fact that $\binom{n}{k_1, k_2} = \binom{n}{k_1}$, and $k_2 = n - k_1$, it becomes: $(x+y)^n = \sum_{\substack{k_1 + k_2 = n \\ k_i \text{ nonnegative integer}}} \binom{n}{k_1} x^{k_1} y^{n-k_1}$. The proof is completed by noticing that this last summation is equivalent to $\sum_{k_1=0}^{n} \binom{n}{k_1} x^{k_1} y^{n-k_1} = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

EFR 5.14: By the multinomial theorem, the full term is $\binom{14}{6,3,3,2}(2a)^6(-3b)^3(4c)^3(-d)^2$. Working out the coefficient, we get $-123{,}986{,}903{,}040$.

EFR 5.15: Part (a): Following the suggestion, we view the problem as the equivalent problem of counting how many ways we can distribute 12 identical balls into 4 different urns: Urn 1, Urn 2, Urn 3, and Urn 4. Thus, x_i represents the number of balls that are placed in Urn i. By Theorem 5.10, it follows that the number of ways this can be done (= the number of nonnegative integer solutions to the given equation) is $\binom{12+(4-1)}{4-1} = \binom{15}{3} = 455$.

Part (b): We introduce new variables $y_i = x_i + 1$, which will represent positive integers, and let the x_i's still represent nonnegative integers. Thus, the number of solutions of the equation $y_1 + y_2 + y_3 + y_4 = 12$ (in positive integers) is the same as the number of solutions of the equation $(x_1+1) + (x_2+1) + (x_3+1) + (x_4+1) = 12$ or $x_1 + x_2 + x_3 + x_4 = 8$ (in nonnegative integers). By the method developed in the solution of Part (a), this latter equation has $\binom{8+3}{3} = \binom{11}{3} = 165$ solutions, and so this is the number of positive integer solutions of the given equation.

EFR 5.16: The number of terms in the sum on the right-hand side of (8) (the multinomial theorem) is just the number of nonnegative integer solutions of the equation $k_1 + k_2 + \cdots + k_r = n$. By the method of the solution to Part (a) of the preceding exercise for the reader, this number is $\binom{n+r-1}{r-1}$.

EFR 5.17: Part (a): Using (8), we obtain $x^2 + x^3 + x^4 + \cdots = x^2(1 + x + x^2 + \cdots) = x^2/(1-x)$.
Part (b): Substituting $x \mapsto 2x$ in (9), it becomes $e^{2x} = 1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!} + \cdots$, so the given generating function is $(1-x)e^{2x} = 1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!} + \cdots$.

EFR 5.18: Using Definition 5.6, for $k > 0$, we may write:

$$\begin{aligned}\binom{a}{k} = \binom{-n}{k} &= \frac{(-n)(-n-1)(-n-2)\cdots(-n-k+1)}{k!}\\ &= (-1)^k \frac{(n)(n+1)(n+2)\cdots(n+k-1)}{k!}\\ &= (-1)^k \frac{(n+k-1)!}{(n-1)!k!} = (-1)^k C(n+k-1,k).\end{aligned}$$

EFR 5.19: This follows directly from the generalized binomial theorem with $a = -N$, along with the result of Exercise for the Reader 5.18 (with k changed to n, and n changed to N).

EFR 5.20: With $a = 1/2$, (10) becomes $(1+x)^{1/2} = \sum_{n=0}^{\infty}\binom{1/2}{n}x^n$. Using Definition 5.6, for $k > 1$ we may write:

$$\begin{aligned}\binom{1/2}{k} &= \frac{(1/2)(1/2-1)(1/2-2)\cdots(1/2-k+1)}{k!} = \frac{(1/2)(-1/2)(-3/2)\cdots([3-2k]/2)}{k!}\\ &= \frac{(-1)^{k-1}}{2^k}\frac{(2k-3)(2k-5)\cdots3\cdot1}{k!} = \frac{(-1)^k}{2^k}\frac{(2k-3)(2k-5)\cdots3\cdot1\cdot(-1)}{k!}.\end{aligned}$$

The last manipulation was done because the final formula is also valid when $k = 1$. Putting this all together, we may write:

$$(1+x)^{1/2} = 1 + x/2 - x^2/8 + x^3/16 - 5x^4/128 + \cdots = 1 + \sum_{n=1}^{\infty}\frac{(-1)^n}{2^n}\frac{(2n-3)(2n-5)\cdots3\cdot1\cdot(-1)}{n!}x^n.$$

Substituting $x \mapsto x/2$ in this expansion leads us to the desired expansion:

$$\sqrt{1+x/2} = 1 + x/4 - x^2/16 + x^3/32 - 5x^4/256 + \cdots = 1 + \sum_{n=1}^{\infty}\frac{(-1)^n}{4^n}\frac{(2n-3)(2n-5)\cdots3\cdot1\cdot(-1)}{n!}x^n.$$

EFR 5.21: We introduce the generating function for the sequence: $F(x) = \sum_{n=0}^{\infty} a_n x^n$. We multiply both sides of the recurrence relation by x^n, and then take the formal (infinite) sum of both sides in the range $n \ge 1$ where the recurrence is valid:

$$a_n = 3a_{n-1} - 1 \ (n \ge 1) \Rightarrow \sum_{n=1}^{\infty} a_n x^n = 3\sum_{n=1}^{\infty} a_{n-1}x^n - \sum_{n=1}^{\infty} 1\cdot x^n.$$

In the same fashion as in the solution of Example 5.22, this equation translates into the following algebraic equation for the generating function:

$$F(x)-1=3xF(x)-\frac{1}{1-x}+1 \Rightarrow (1-3x)F(x)=2-\frac{1}{1-x}=\frac{1-2x}{1-x} \Rightarrow F(x)=\frac{1-2x}{(1-x)(1-3x)}.$$

The partial fractions expansion of the right side is:

$$\frac{1-2x}{(1-x)(1-3x)}=\frac{A}{1-x}+\frac{B}{1-3x}.$$

To determine the constants A and B on the right side, we first multiply both sides of the equation by the denominator on the right to obtain:

$$1-2x=A(1-3x)+B(1-x).$$

If we substitute x = 1 into this equation, we obtain $A=1/2$, and substituting x = 1/3 produces $B=1/2$. The original equation now gives:

$$F(x)=\frac{1}{2}\left\{\frac{1}{1-x}+\frac{1}{1-3x}\right\}.$$

Each of the terms on the right can easily be expanded using (8) (a special case of (12)):

$$F(x)=F(x)=\frac{1}{2}\left\{\frac{1}{1-x}+\frac{1}{1-3x}\right\}=(1/2)\sum_{n=0}^{\infty}x^n+(1/2)\sum_{n=0}^{\infty}(3x)^n=(1/2)\sum_{n=0}^{\infty}(3^n+1)x^n.$$

Thus we have found that $a_n=(3^n+1)/2$.

EFR 5.22: Following the method of the Examples 5.22 and 5.23, we let $F(x)=\sum_{n=0}^{\infty}a_nx^n$ be the generating function for the given sequence, multiply both sides of the recurrence relation by x^n, and then take the formal (infinite) sum of both sides in the range $n\ge 1$ (where the recurrence is valid):

$$a_n=2a_{n-1}+3^n \quad (n\ge 1) \Rightarrow \sum_{n=1}^{\infty}a_nx^n=2\sum_{n=1}^{\infty}a_{n-1}x^n+\sum_{n=1}^{\infty}3^nx^n.$$

The first two sums we have already seen, while the third sum on the right can be converted using the expansion (8) with the substitution $x\mapsto x/2$:

$$\sum_{n=1}^{\infty}3^nx^n=\sum_{n=0}^{\infty}(3x)^n-1=\frac{1}{1-3x}-1.$$

Hence, the preceding power series equation transforms into the following algebraic equation for the generating function:

$$F(x)-1=2xF(x)+\frac{1}{1-3x}-1 \Rightarrow F(x)=\frac{1}{(1-2x)(1-3x)}.$$

The partial fractions expansion will have the form:

$$F(x)=\frac{1}{(1-2x)(1-3x)}=\frac{A}{1-2x}+\frac{B}{1-3x}.$$

To determine the constants A, B, we first clear out all denominators:

$$1 = A(1-3x) + B(1-2x).$$

Substituting $x = 1/2$ yields $A = -2$, and substituting $x = 1/3$ yields $B = 3$. Thus we have determined the partial fractions expansion of the generating function to be:

$$F(x) = \frac{-2}{1-2x} + \frac{3}{1-3x}.$$

Applying the expansion (8) twice, we obtain

$$F(x) = \frac{-2}{1-2x} + \frac{3}{1-3x} = -2\sum_{n=0}^{\infty}(2x)^n + 3\sum_{n=0}^{\infty}(3x)^n = \sum_{n=0}^{\infty}(3^{n+1} - 2^{n+1})x^n.$$

We have thus arrived at the following closed formula for the given recursively defined sequence: $a_n = 3^{n+1} - 2^{n+1}$.

<u>EFR 5.23:</u> Although the Fibonacci recurrence relation involves two prior terms, the generating function approach will be the same. We do make one small cosmetic adjustment because it is convenient to have our sequence indices begin at $n = 0$. We introduce the shifted sequence $e_n = f_{n+1}$, which is defined for all $n \ge 0$, and inherits the following recursive definition from that of f_n:

$$\begin{cases} e_0 = 1, e_1 = 1, \\ e_n = e_{n-1} + e_{n-2}\ (n \ge 2). \end{cases}$$

It suffices to obtain an explict formula for e_n, since the corresponding formula for the Fibonacci sequence and then be read off from the relation: $f_n = e_{n-1}$.

We let $F(x) = \sum_{n=0}^{\infty} e_n x^n$ be the generating function for sequence e_n, multiply both sides of the recurrence relation by x^n, and then take the formal (infinite) sum of both sides in the range $n \ge 2$ (where the recurrence is valid):

$$e_n = e_{n-1} + e_{n-2}\ (n \ge 2) \Rightarrow \sum_{n=2}^{\infty} e_n x^n = \sum_{n=2}^{\infty} e_{n-1} x^n + \sum_{n=2}^{\infty} e_{n-2} x^n.$$

Using the first two coefficients: $e_0 = 1, e_1 = 1$, we may transform each of these three sums in terms of the generating function:

$$\sum_{n=2}^{\infty} e_n x^n = F(x) - x - 1, \quad \sum_{n=2}^{\infty} e_{n-1} x^n = x\sum_{n=2}^{\infty} e_{n-1} x^{n-1} = x(F(x) - 1), \quad \sum_{n=2}^{\infty} e_{n-2} x^n = x^2 \sum_{n=2}^{\infty} e_{n-2} x^{n-2} = x^2 F(x).$$

The preceding power series equation thus transforms into the following algebraic equation for the generating function:

$$F(x) - x - 1 = x(F(x) - 1) + x^2 F(x) \Rightarrow F(x) = \frac{1}{1 - x - x^2}.$$

The denominator on the right has two real roots: $x = (-1 \pm \sqrt{5})/2$, which we temporarily will denote by r_+ and r_-. The partial fractions decomposition of $F(x)$ will take the form:

$$F(x) = \frac{1}{1 - x - x^2} = \frac{A}{1 - x/r_+} + \frac{B}{1 - x/r_-}.$$

To determine the constants A, B, we first clear out all denominators:

$$1 = A(1 - x/r_-) + B(1 - x/r_+).$$

Substituting $x = r_+$ yields $A = \dfrac{1}{1 - r_+/r_-} = \dfrac{r_-}{r_- - r_+} = \dfrac{r_-}{-\sqrt{5}}$, and in a similar fashion, substituting $x = r_-$ yields $B = \dfrac{r_+}{\sqrt{5}}$. Thus we have determined the partial fractions expansion of the generating function to be:

$$F(x) = \frac{1}{\sqrt{5}}\left\{\frac{-r_-}{1 - x/r_+} + \frac{r_+}{1 - x/r_-}\right\}.$$

Applying the expansion (8) twice, we obtain

$$F(x) = \frac{1}{\sqrt{5}}\left\{\frac{-r_-}{1 - x/r_+} + \frac{r_+}{1 - x/r_-}\right\} = \frac{1}{\sqrt{5}}\left\{-r_- \sum_{n=0}^{\infty} (x/r_+)^n + r_+ \sum_{n=0}^{\infty} (x/r_-)^n\right\} = \frac{1}{\sqrt{5}} \sum_{n=0}^{\infty} (-r_-(1/r_+)^n + r_+(1/r_-)^n)x^n.$$

We have thus arrived at a closed formula for the given recursively defined sequence, which we can simplify by noting that $r_+ r_- = -1$:

$$e_n = \frac{1}{\sqrt{5}}\left(-\frac{r_-}{r_+^{\,n}} + \frac{r_+}{r_-^{\,n}}\right) = \frac{1}{\sqrt{5}}\left(\frac{-r_-^{\,n+1} + r_+^{\,n+1}}{r_+^{\,n} r_-^{\,n}}\right) = \frac{1}{\sqrt{5}}\left(\frac{-[(-\sqrt{5} - 1)/2]^{n+1} + [(\sqrt{5} - 1)/2]^{n+1}}{(-1)^n}\right)$$
$$= \frac{1}{\sqrt{5}}\left(\frac{\sqrt{5} + 1}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1 - \sqrt{5}}{2}\right)^{n+1}.$$

Finally, since $f_n = e_{n-1}$, this translates to the following explicit formula for the Fibonacci sequence:

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{\sqrt{5} + 1}{2}\right)^{n} - \frac{1}{\sqrt{5}}\left(\frac{1 - \sqrt{5}}{2}\right)^{n},$$

which was first introduced in Chapter 3.

EFR 5.24: For both questions being asked, we may take the generating function for the number of \$1 bills drawn to be $F_O(x) = 1(= x^0) + x + x^2 + x^3 + x^4 + x^5 + x^6$. Since there are only five \$10 bills, the corresponding generating function for the number of \$10 bills drawn is $F_T(x) = 1 + x + x^2 + x^3 + x^4 + x^5$, and since there are only two \$100 bills, the generating function for the number of these bills drawn is $F_H(x) = 1 + x + x^2$. The generating function for the both questions is the product of these three:

$$F_O(x) \cdot F_T(x) \cdot F_H(x) = (1 + x + x^2 + x^3 + x^4 + x^5 + x^6) \cdot (1 + x + x^2 + x^3 + x^4 + x^5) \cdot (1 + x + x^2).$$

The number of ways that four bills can be drawn is the coefficient of x^4 in this product, which can be computed to be 12, and the number of ways that six bills can be drawn is the coefficient of x^6 in this product, which can be computed to be 17.

EFR 5.25: As was pointed out in Example 5.28, any winning coalition must contain either V_1 or V_2; below we consider only the winning coalitions:

1. Four winning coalitions of the form $\{V_1, V_2\} \cup S$, where S is any subset of $\{V_4, V_5\}$:

Critical Voters: V_1, V_2

2. Four winning coalitions of the form $\{V_1, V_3\} \cup S$, where S is any subset of $\{V_4, V_5\}$:

Critical Voters: V_1, V_3

3. Four winning coalitions of the form $\{V_2, V_3\} \cup S$, where S is any subset of $\{V_4, V_5\}$:

Critical Voters: V_2, V_3

4. Four winning coalitions of the form $\{V_1, V_2, V_3\} \cup S$, where S is any subset of $\{V_4, V_5\}$:

Critical Voters: None

It follows that $c(V_1) = c(V_2) = c(V_3) = 4$, $c(V_4) = c(V_5) = 0$, $T = 12$, so the Banzhaf indices of V_1, V_2, V_3 are each 1/3, and those of V_4, V_5 are zero.

EFR 5.26: Part (a): Rather than going through all $2^6 - 1 = 65$ coalitions, this example is small enough to take advantage of its special structure. Clearly any winning coalition must include at least two of V_1, V_2, V_3; and conversely any coalition containing two of these three voters will be winning. Here is a summary count of these winning coalitions along with critical voters:

1. Eight winning coalitions of the form $\{V_1, V_2\} \cup S$, where S is any subset of $\{V_4, V_5, V_6\}$:

Critical Voters: V_1, V_2

2. Eight winning coalitions of the form $\{V_1, V_3\} \cup S$, where S is any subset of $\{V_4, V_5, V_6\}$:

Critical Voters: V_1, V_3

3. Eight winning coalitions of the form $\{V_2, V_3\} \cup S$, where S is any subset of $\{V_4, V_5, V_6\}$:

Critical Voters: V_2, V_3

4. Eight winning coalitions of the form $\{V_1, V_2, V_3\} \cup S$, where S is any subset of $\{V_4, V_5, V_6\}$:

Critical Voters: None

It follows that $c(V_1) = c(V_2) = c(V_3) = 16$, $c(V_4) = c(V_5) = c(V_6) = 0$, $T = 48$, so the Banzhaf indices of V_1, V_2, V_3 are each 1/3, and those of V_4, V_5, V_6 are zero.

Part (b): None of the voters are dictators or have veto power; but V_4, V_5, V_6 are dummies.

EFR 5.27: *Step 1:* The total weight is $W = 5 + 2 + 1 = 8$. The number of voters is $N = 3$.

Set $a_0 = a_W = 1$. Set $a_0^{(0)} = 1$, and $a_1^{(0)} = a_2^{(0)} = \cdots = a_8^{(0)} = 0$.

Note: $a_i^{(0)}$ are the coefficients of $F_0(x) = 1$.

Step 2: (*Compute the* $a_i^{(j)}$ *'s*)

FOR index $j = 1$, FOR index $i = 0$ TO $i = W - 1$, Set $a_i^{(1)} = a_i^{(0)} + a_{i-5}^{(0)}$: (since $w_1 = 5$)

$$a_0^{(1)} = a_0^{(0)} + a_{0-5}^{(0)} = 1 + 0 = 1, a_1^{(1)} = a_1^{(0)} + a_{1-5}^{(0)} = 0 + 0 = 0, \cdots, a_5^{(1)} = a_5^{(0)} + a_{5-5}^{(0)} = 0 + 1 = 1,$$
$$a_6^{(1)} = a_6^{(0)} + a_{6-5}^{(0)} = 0 + 0 = 0, a_7^{(1)} = a_7^{(0)} + a_{7-5}^{(0)} = 0 + 0 = 0.$$

Note: $a_i^{(1)}$ are the coefficients of $F_1(x) = 1 + x^{w_1} = 1 + x^5$.

FOR index $j = 2$, FOR index $i = 0$ TO $i = W - 1$, Set $a_i^{(2)} = a_i^{(1)} + a_{i-2}^{(1)}$: (since $w_2 = 2$)

$$a_0^{(2)} = a_0^{(1)} + a_{0-2}^{(1)} = 1 + 0 = 1, a_1^{(2)} = a_1^{(1)} + a_{1-2}^{(1)} = 0 + 0 = 0, a_2^{(2)} = a_2^{(1)} + a_{2-2}^{(1)} = 0 + 1 = 1,$$
$$a_3^{(2)} = 0 + 0 = 0, a_4^{(2)} = 0 + 0 = 0, a_5^{(2)} = 1 + 0 = 1, a_6^{(2)} = 0 + 0 = 0, a_7^{(2)} = 0 + 1 = 1.$$

Note: $a_i^{(2)}$ are the coefficients of $F_2(x) = (1 + x^{w_1})(1 + x^{w_2}) = (1 + x^5)(1 + x^2) = 1 + x^2 + x^5 + x^7$.

FOR index $j = 3$, FOR index $i = 0$ TO $i = W - 1$, Set $a_i^{(3)} = a_i^{(2)} + a_{i-1}^{(2)}$: (since $w_3 = 1$)

$$a_0^{(3)} = 1 + 0 = 1, a_1^{(3)} = 0 + 1 = 1, a_2^{(3)} = 1 + 0 = 1, a_3^{(3)} = 0 + 1 = 1, a_4^{(3)} = 0 + 0 = 0,$$
$$a_5^{(3)} = 1 + 0 = 1, a_6^{(3)} = 0 + 1 = 1, a_7^{(3)} = 1 + 0 = 1.$$

Note: $a_i^{(3)}$ are the coefficients of $F_3(x)=(1+x^5)(1+x^2)(1+x)=1+x+x^2+x^3+x^5+x^6+x^7$.

Step 3: (*Record the* a_i *'s*) Set $a_i = a_i^{(N)}$ ($i=1$ TO $i=W-1$)

$$a_1 = a_1^{(3)}=1, a_2=a_2^{(3)}=1, a_3=1,\ a_4=0, a_5=1, a_6=1, a_7=1.$$

Step 4: (*Compute the* σ_n *'s*)

FOR index $n=1$, $b_i = a_i - b_{i-5}$ ($i=0$ TO $i=q-1=6$)

$$b_0=a_0-b_{0-5}=1, b_1=a_1-b_{1-5}=1, b_2=1, b_3=1, b_4=0, b_5=0, b_6=0$$

$$\sigma_1=\sum\nolimits_{i=q-w_1}^{q-1} b_i = b_2+b_3+b_4+b_5+b_6=2$$

FOR index $n=2$, $b_i = a_i - b_{i-2}$ ($i=0$ TO $i=q-1=6$)

$$b_0=a_0-b_{0-2}=1, b_1=a_1-b_{1-2}=1, b_2=0, b_3=0, b_4=0, b_5=1, b_6=1$$

$$\sigma_2=\sum\nolimits_{i=q-w_2}^{q-1} b_i = b_4+b_5+b_6=2$$

FOR index $n=3$, $b_i = a_i - b_{i-1}$ ($i=0$ TO $i=q-1=6$)

$$b_0=a_0-b_{0-1}=1, b_1=a_1-b_{1-1}=0, b_2=1, b_3=0, b_4=0, b_5=1, b_6=0$$

$$\sigma_3=\sum\nolimits_{i=q-w_3}^{q-1} b_i = b_6=0$$

Step 5: (*Compute the Banzhaf indices*)

Set $T=\sum\nolimits_{n=1}^{N}\sigma_n = 2+2+0=4$

Banzhaf index of $V_1=\sigma_1/T=2/4=0.5$, Banzhaf index of $V_2=\sigma_2/T=2/4=0.5$,
Banzhaf index of $V_3=\sigma_3/T=0/4=0$.

ANSWERS/BRIEF SOLUTIONS TO ODD-NUMBERED EXERCISES

CHAPTER 5: Section 5.1: **#1.** (a) $10\cdot 26^3\cdot 10^3 = 175,760,000$ (b) $10\cdot 26\cdot 25\cdot 24\cdot 9\cdot 8\cdot 7 = 78,624,000$ **#3.** $4\cdot 4 = 16$ **#5.** (a) $5\cdot 4\cdot 3\cdot 2\cdot 1 = 120$ (b) $2\cdot 1\cdot 3\cdot 2\cdot 1 = 12$ (c) $2\cdot 4\cdot 3\cdot 2\cdot 1 = 48$ **#7.** (a) $12\cdot 11\cdot 10 = 1320$ (b) $1320 - 9\cdot 8\cdot 7 = 816$ (c) $7\cdot 6\cdot 5 + (10\cdot 9\cdot 8 - 7\cdot 6\cdot 5) + (9\cdot 8\cdot 7 - 7\cdot 6\cdot 5) = 1014$ (d) $3\cdot 2\cdot 10 + 10\cdot 9\cdot 8 = 780$ **#9.** (a) $5^7 = 78,125$ (b) $2^3\cdot 5^4 = 5000$ (c) 0 **#11.** (a) $[(3\cdot 2)/2]\cdot 4^2 = 48$ (b) $[(3\cdot 2)/2]\cdot[(4\cdot 3)/2]^2 = 108$ **#13.** (a) $62^5 + 62^6 + 62^7 = 3,579,330,974,624$ (b) $\sum_{j=5}^{7}[62^j - (52^j + 10^j)] = 2,531,097,358,400$ (c) $62^5 + 62^6 + 62^7 - (52^5 + 52^6 + 52^7) - 2\cdot(36^5 + 36^6 + 36^7) + 2\cdot(26^5 + 26^6 + 26^7) + (10^5 + 10^6 + 10^7) =$ 2,386,621,947,840 (d) $3\cdot 8\cdot 62^2 + [4\cdot 8\cdot 62^3 - 8^3] + [5\cdot 8\cdot 62^4 - 3\cdot 8^2\cdot 62] = 598,760,224$ **#15.** (a) 3257 (b) 3166 (c) For part (a):

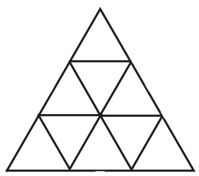
```
set count = 0;
for n =  1 TO n = 5999
   set a = n/3, b = n/5, c = n/7
   if floor(a)==a OR floor(b)==b OR floor(c)==c
       UPDATE count = count+1
   end
end
OUTPUT count
```

For the corresponding program for part (b) change 5 to 11, and replace the three "OR"s with "AND"s and three "=="'s with "~="'s. **#17.** (a) 3506 (b) The program is similar to the one for #15(a); just add one more letter d = n/11, and one more (OR) condition for d. **#19.** (a) 1598 (b) 830 (c) The program for part (a) is similar to that of parts (a) and (b) of Exercise 15. By first using the complement principle, the problem of part (b) is reduced to one like in part (a), and can be programmed accordingly.

```
set count = 0;
for n =  1 TO n = 3999
   set a = n/2, b = n/3, c = n/5, d = n/7, e = n/11
   if floor(a)~=a AND floor(b)~=b AND floor(c)~=c
           AND floor(d)~=d AND floor(c)~=c
       UPDATE count = count+1
   end
end
OUTPUT count
```

Alternatively, a program can be written directly as shown (left):

#21. (a) Rounded to the nearest inch, there are exactly 20 possible heights between 5-0 and 6-7 (these are the pigeonholes), therefore, in a group of 21 men in this height range, at least two must be the same height. (b) We use the generalized pigeonhole principle with the four suits serving as the pigeons. The smallest integer N for which $\lceil N/4 \rceil = 3$ is $N = 9$, and so this is the smallest number of cards to guarantee at least three will have the same suit. With only 8 cards, we could have 2 of each suit. **#23** (a) The pigeonholes are the nine smaller equilateral triangles (each having side length 1/3) without their vertices, as shown in the figure. If ten points are put in the interior of the big triangle, then at least two must lie in the same pigeonhole and therefore have distance between them less than 1/3. (b) If we take nine points to be the vertices of the smaller triangles that lie on the edges of the larger triangle, then any two of these will be at a distance of at least 1/3.



Section 5.2: **#1.** (a) CAT, CTA, ACT, ATC, TAC, TCA (b) AB, BA, AC, CA, AD, DA, AE, EA, BC, CB, BD, DB, BE, EB, CD, DC, CE, EC, DE, ED (c) {A,B}, {A,C}, {A,D}, {A,E}, {B,C}, {B,D}, {B,E}, {C,D}, {C,E}, {D,E} **#3.** (a) 6 (b) 1 (c) 132,600 (d) 15,504 (e) 5 (f) 1 **#5.** (a) $C(12,3) = 220$ (b) $\sum_{k=0}^{3} C(12,k) = 299$ **#7.** (a) $C(12,4) = 495$ (b) $P(12,7) = 3{,}991{,}680$ **#9.** (a) $C(8,5) = 56$ (b) $C(8,5)\cdot 3^5 = 13{,}608$ (c) $C(6,3)\cdot 3^5 + C(6,5)\cdot 3^5 = 6318$ (d) $C(6,3)\cdot 3^4 + C(6,5)\cdot 3^5 = 3078$ **#11.** (a) $C(26,2) = 325$ (b) $C(15,2) + C(11,2) = 160$ (c) $C(11,2) + 11\cdot 15 = 220$ **#13.** (a) $C(39,4) + C(39,3)\cdot C(13,1) = 201{,}058$ (b) $13^4 = 28{,}561$ (c) $C(52,4) - C(13,1)^4$(*all different suits*) $= 242{,}164$ **#15.** (a) C(8,5) = 56; $C(4,3)\cdot C(4,2) + C(4,4)\cdot C(4,1) = 24 + 4 = 28$ **#17.** (a) 6!2! = 1440 (b) 5!3! = 720 (c) 7!/3 = 1680 (d) 5! = 120 (e) 4! = 24 **#19.** (a) 10,897,286,400 (b) $P(14,9) + P(15,9) = 2{,}542{,}700{,}160$ **#21.** (a) $x^7 + 7x^6z + 21x^5z^2 + 35x^4z^3 + 35x^3z^4 + 21x^2z^5 + 7xz^6 + z^7$ (b) $3125x^5 + 3125x^4y^3 + 1250x^3y^6 + 250x^2y^9 + 25xy^{12} + y^{15}$ (c) $\binom{5}{3}3^3(-4)^2 = 4320$ **#25.** (a) $16x^4 - 64x^3y + 160x^3z + 96x^2y^2 - 480x^2yz + 600x^2z^2 - 64xy^3 + 480xy^2z - 1200xyz^2 + 1000xz^3 + 16y^4 - 160y^3z + 600y^2z^2 - 1000yz^3 + 625z^4$ (c) $\binom{20}{4,6,4,6}2^6 3^4 = 42{,}237{,}882{,}086{,}400$

#27.(a) $\binom{4}{2,2} = \binom{4}{2} = 6$ (b) $\binom{8}{3,2,2,1} = 1680$ (c) $\binom{9}{3,2,1,1,1,1} = 30{,}240$ (d) $\binom{10}{3,2,2,2,1} = 75{,}600$

#29. (a) $\binom{9}{3,2,4} = 1260$ (b) $3\cdot 3\cdot 3 - 1 = 26$ (can't have ggg) (c) $26 + 3^2 + 3 = 38$

#31. (a) Let x_A be the number of \$500 incremements placed in Fund A, and similarly for x_B, x_C. The number of permissible fund allocations is the number of nonnegative integer solutions of the equation $x_A + x_B + x_C = 30$; and by Theorem 5.10, this number is $\binom{30+(3-1)}{3-1} = \binom{32}{2} = 496.$

(b) Here we seek the number of solutions of the integer equation above with the additional constraints that $x_A, x_C \ge 5$. If we define $y_A = x_A - 5, y_C = x_C - 5,$ the problem is transferred to counting the number of nonnegative integer solutions of the equation $y_A + x_B + y_C = 20$. By Theorem 5.10, it follows that this number is $\binom{20+(3-1)}{3-1} = \binom{22}{2} = 231.$

#33. Think of placing the 12 donuts in an arbitrary dozen into 8 bins, according to the type of donut. By Theorem 5.10, it follows that the number of dozens of donuts is given by $\binom{12+(8-1)}{8-1} = \binom{19}{7} = 50{,}388.$

#35. (a) Let S be a set of n (distinct) objects. Any subset A of S containing k objects (i.e., a k-combination) naturally corresponds to a subset of S containing $n-k$ objects, namely its complement $\sim A = S \sim A$. Since this correspondence is one-to-one, it follws that the number of k-combinations, namely $\binom{n}{k}$, must equal the number of $(n-k)$-combinations, namely $\binom{n}{n-k}$. A noncombinatorial proof is simpler (but less revealing) due to the symmetry in the definition of the binomial coeffients:

$$\binom{n}{n-k}=\frac{n!}{(n-k)!(n-[n-k])!}=\frac{n!}{(n-k)!k!}=\frac{n!}{k!(n-k)!}=\binom{n}{k}.$$

(b) Answer: If n is even, $k = n/2$; if n is odd, $k=\lfloor n/2\rfloor$ and $\lceil n/2\rceil$.

Proof: Suppose that $0<k\le n/2$. Since this implies that $n\ge 2k$, we obtain:

$$\binom{n}{k}\Big/\binom{n}{k-1}=\frac{(k-1)!(n-k+1)!}{k!(n-k)!}=\frac{n-k+1}{k}\ge\frac{2k-k+1}{k}=\frac{k+1}{k}>1.$$

This proves that that the left half of the binomial coefficients are an increasing sequence:

$$\binom{n}{0}<\binom{n}{1}<\binom{n}{2}<\cdots<\binom{n}{\lfloor n/2\rfloor}.$$

From the symmetry result of part (a), it follows that the corresponding right half of the binomial coefficient sequence is decreasing:

$$\binom{n}{\lfloor n/2\rfloor+1}>\binom{n}{\lfloor n/2\rfloor+2}>\binom{n}{\lfloor n/2\rfloor+3}>\cdots>\binom{n}{n}.$$

In case n is even, there is a single middle coefficient $\binom{n}{\lfloor n/2\rfloor}=\binom{n}{n/2}$ that is larger than any other.

In case n is odd, there are two equal largest middle coefficients $\binom{n}{\lfloor n/2\rfloor}=\binom{n}{\lfloor n/2\rfloor+1}$.□

#37. (a) 0 (b) 6! (c) $\mathrm{C}(6,2)\cdot 5!$ (d) 2^7-2

#43. We consider the combinatorial problem of choosing a team of k people with a designated team leader from a group of n people. By the multiplication principle, the total number of ways to form such a team is

(the number of ways to choose a subset k people from n people) ·

(the number of ways of choosing a team leader from a team of k people)

$$=\binom{n}{k}\cdot k=k\binom{n}{k}.$$

This number can also be computed as

(the number of ways to choose a leader from a group of n people) ·

(the number of ways of choosing the $k-1$ non leaders from the remaining $n-1$ people)

$$=n\binom{n-1}{k-1}.$$

It follows that $k\binom{n}{k}=n\binom{n-1}{k-1}$.

Section 5.3:

#1. (a) $1-x+x^2-x^3+\cdots=\sum_{n=0}^{\infty}(-1)^n x^n$ (b) $6+4x^2+2x^4+x^6$

#3. (a) $1-x+x^2-x^3+\cdots-x^9+x^{10}=\sum_{n=0}^{10}(-x)^n=\frac{1-(-x)^{11}}{1-(-x)}=\frac{1+x^{11}}{1+x}$. (By use of Proposition 3.5.)

(b) Using the binomial theorem, we may write:

$C(10,0)+C(10,1)x+C(10,2)x^2+\cdots+C(10,10)x^{10}=\sum_{n=0}^{10}C(10,n)x^n=\sum_{n=0}^{10}C(10,n)x^n1^{10-n}=(x+1)^{10}$.

(c) $20-40x+80x^2-160x^3+320x^4=20\sum_{n=0}^{5}(-2x)^n=20\cdot\frac{1-(-2x)^5}{1-(-2x)}=20\cdot\frac{1+32x^5}{1+2x}$. (By use of Theorem 3.5.)

(d) Using the binomial theorem, we may write:

$$C(5,0)x^2 + C(5,1)x^3 + C(5,2)x^4 + C(5,3)x^5 + C(5,4)x^6 + C(5,5)x^7$$
$$= x^2\sum_{n=0}^{5} C(5,n)x^n = x^2\sum_{n=0}^{5} C(5,n)x^n 1^{5-n} = x^2(x+1)^5.$$

#5. (a) $\sum_{n=0}^{\infty}(-x)^n \underset{\text{By (8)}}{=} \frac{1}{1-(-x)} = \frac{1}{1+x}$ (b) $\sum_{n=0}^{\infty}\frac{(-2)^n}{n!}x^n = \sum_{n=0}^{\infty}\frac{(-2x)^n}{n!} \underset{\text{By (9)}}{=} e^{-2x}$

(c) $\sum_{n=0}^{\infty}(-x)^n + x - 5x \underset{\text{By (8)}}{=} \frac{1}{1-(-x)} + x - 5x = \frac{1}{1+x} - 4x$

(d) $\sum_{n=0}^{\infty}\frac{1}{(n+2)!}x^n = x^{-2}\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!} = x^{-2}\sum_{n=2}^{\infty}\frac{x^n}{n!} \underset{\text{By (9)}}{=} \frac{e^x - 1 - x}{x^2}$

#7. (a) Using the binomial theorem: $x^3(x+5)^4 = x^3\sum_{n=0}^{4}C(4,n)x^n 5^{4-n} = \sum_{n=0}^{4}[C(4,n)5^{4-n}]x^{n+3}$, so the sequence is: $a_n = C(4,n)5^{4-n}$ if $n = 3,4,5,6,7$ and $a_n = 0$ if $n = 0,1,2$.

(b) Using the binomial theorem:

$$(1-x)^3 - x^5 = C(3,0) + C(3,1)(-x) + C(3,2)(-x)^2 + C(3,3)(-x)^3 - x^5 = 1 - 3x + 3x^2 - x^3 - x^5,$$

so the sequence is: $a_0 = 1,\ a_1 = 3,\ a_2 = -3,\ a_3 = -1,\ a_4 = 0,\ a_5 = -1.$

(c) $1/(1+3x) = 1/(1-(-3x)) \underset{\text{By (8)}}{=} \sum_{n=0}^{\infty}(-3x)^n$, so the sequence is $a_n = (-3)^n$ $(n = 0,1,2,\cdots)$

(d) From the expansion: $1/(1+x) - x/(1-2x) = 1/(1-(-x)) - x\cdot 1/(1-(2x)) \underset{\text{By (8)}}{=} \sum_{n=0}^{\infty}(-x)^n - x\sum_{n=0}^{\infty}(-2x)^n = 1 + \sum_{n=0}^{\infty}[(-1)^n - (-2)^n]x^n$, the sequence is: $a_0 = 1,\ a_n = (-1)^n - (-2)^n$ $(n = 1,2,3,\cdots)$.

(e) We clear out denominators in the partial fractions decomposition: $1/[(1+x)(1-2x)] = A/(1+x) + B/(1-2x)$, to obtain: $1 = A(1-2x) + B(1+x)$. Substituting $x = -1$ yields $A = 1/3$, and substituting $x = 1/2$ yields $B = 2/3$. Next, $[1/(1+x) + 2/(1-2x)] \underset{\text{By (8)}}{=} (1/3)\sum_{n=0}^{\infty}(-x)^n + (2/3)\sum_{n=0}^{\infty}(2x)^n = \sum_{n=0}^{\infty}[(-1)^n/2 + 2\cdot 2^n/3]x^n$, so the sequence is $a_n = [(-1)^n + 2\cdot 2^n]/3\ (n = 0,1,2,\cdots)$

(f) Using (9), we may write: $e^{2x}(1-x^2) = \sum_{n=0}^{\infty}[x^n/n!](1-x^2) = \sum_{n=0}^{\infty}[2^n/n!]x^n - \sum_{n=0}^{\infty}[2^n/n!]x^{n+2}$ $= \sum_{n=0}^{\infty}[2^n/n!]x^n - \sum_{n=2}^{\infty}[2^{n-2}/(n-2)!]x^n$, so the sequence is $a_0 = 1, a_1 = 2, a_n = 2^n/n! - 2^{n-2}/(n-2)!$ $(n = 2,3,4,\cdots)$

#9. (a) 5 (b) 3 (c) –4 (d) 14

#11. (a) $\sum_{n=0}^{\infty}b_n x^n \triangleq x^3\sum_{n=0}^{\infty}a_n x^n = \sum_{n=0}^{\infty}a_n x^{n+3} = \sum_{n=3}^{\infty}a_{n-3}x^n \Rightarrow b_n = a_{n-3}$ (which tacitly implies $b_0 = b_1 = b_2 = 0$, because of the convention that unassigned coefficients, like $a_{-1}, a_{-2}, \cdots$ are taken to be zero).

(b) $\sum_{n=0}^{\infty}b_n x^n \triangleq (1-x)\sum_{n=0}^{\infty}a_n x^n = \sum_{n=0}^{\infty}a_n x^n - \sum_{n=0}^{\infty}a_n x^{n+1} = \sum_{n=0}^{\infty}[a_n - a_{n-1}]x^n$

$\Rightarrow$ $b_n = a_n - a_{n-1}$ (which tacitly implies $b_0 = a_0$).

(c) $\sum_{n=0}^{\infty}b_n x^n \triangleq \sum_{n=0}^{\infty}a_n x^n \cdot 1/(1-x) \underset{\text{By (8)}}{=} \sum_{n=0}^{\infty}a_n x^n \cdot \sum_{n=0}^{\infty}x^n = \sum_{n=0}^{\infty}[\sum_{k=0}^{n}a_k]x^n$

$\Rightarrow b_n = \sum_{k=0}^{n}a_k.$

#13. (a) –56 (b) –0.02734375

#15. (a) With $a = -1/2$, (10) becomes $(1+x)^{-1/2} = \sum_{n=0}^{\infty}\binom{-1/2}{n}x^n.$ Using Definition 5.6, for $k > 1$ we may write:

$$\binom{-1/2}{k} = \frac{(-1/2)(-1/2-1)(-1/2-2)\cdots(-1/2-k+1)}{k!} = \frac{(-1/2)(-3/2)(-5/2)\cdots([1-2k]/2)}{k!}$$
$$= \frac{(-1)^k}{2^k}\frac{(2k-1)(2k-3)\cdots 3\cdot 1}{k!}.$$

It follows that

$$(1+x)^{-1/2} = 1 - x/2 + 3x^2/8 - 5x^3/16 + 35x^4/128 - \cdots = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^n}\frac{(2n-1)(2n-3)\cdots 3\cdot 1}{n!}x^n.$$

(b) (i) With $a = 7/2$, (10) becomes $(1+x)^{7/2} = \sum_{n=0}^{\infty}\binom{7/2}{n}x^n = 1 + \binom{7/2}{1}x + \binom{7/2}{2}x^2 + \binom{7/2}{3}x^3 + \cdots$

Since $\binom{7/2}{1} = \frac{7/2}{1!} = \frac{7}{2}, \binom{7/2}{2} = \frac{7/2\cdot 5/2}{2!} = \frac{35}{8}, \binom{7/2}{3} = \frac{7/2\cdot 5/2\cdot 3/2}{3!} = \frac{35}{16},$ we may write:

$(1+x)^{7/2} = 1 + 7x/2 + 35x^2/8 + 35x^3/16 + \cdots$

(b) (ii) Using (9) and the expansion obtained in part (a), and then multiplying the two generating functions (according to Definition 5.5), we obtain:

$$e^x/\sqrt{1+x} = e^x\cdot(1+x)^{-1/2} = (1 + x + x^2/2! + x^3/3! + \cdots)\cdot(1 - x/2 + 3x^2/8 - 5x^3/16 + \cdots)$$
$$= 1 + x/2 + 3x^2/8 - x^3/48 + \cdots$$

#17. (a) Let $F(x) = \sum_{n=0}^{\infty} a_n x^n$. $a_n = 2a_{n-1} + 5\ (n \ge 1) \Rightarrow \sum_{n=1}^{\infty} a_n x^n = 2\sum_{n=1}^{\infty} a_{n-1}x^n + 5\sum_{n=1}^{\infty} x^n$

$\Rightarrow F(x) - 3 = 2xF(x) + 5/(1-x) - 5 \Rightarrow F(x) = \dfrac{2x+3}{(1-2x)(1-x)} = \dfrac{A}{1-2x} + \dfrac{B}{1-x}$. To find A and B in this partial fractions expansion, we first clear out the denominators $2x + 3 = A(1-x) + B(1-2x)$. Substituting $x = 1$ gives $B = -5$, and substituting $x = 1/2$ gives $A = 8$. Applying the expansion (8) we obtain: $F(x) = \dfrac{8}{1-2x} - \dfrac{5}{1-x} = 8\sum_{n=0}^{\infty}(2x)^n - 5\sum_{n=0}^{\infty}x^n = \sum_{n=0}^{\infty}(2^{n+3} - 5)x^n$, and hence: $a_n = 2^{n+3} - 5$.

(b) Let $b_n = a_{n+2} (n \ge 0)$, so that $\begin{cases} b_0 = 1 \\ b_n = 3b_{n-1} - 1\ (n \ge 1)\end{cases}$. By the solution of Exercise for the Reader 5.21, we have $b_n = (3^n + 1)/2$, so it follows that $a_n = b_{n-2} = (3^{n-2} + 1)/2\ (n \ge 2)$.

(c) Let $F(x) = \sum_{n=0}^{\infty} a_n x^n$. $a_n = 2a_{n-1} + 3n\ (n \ge 1) \Rightarrow \sum_{n=1}^{\infty} a_n x^n = 2\sum_{n=1}^{\infty} a_{n-1}x^n + 3\sum_{n=1}^{\infty} nx^n \Rightarrow$ $\sum_{n=1}^{\infty} a_n x^n = 2x\sum_{n=1}^{\infty} a_{n-1}x^{n-1} + 3x\sum_{n=1}^{\infty} nx^{n-1}$. Using expansions (8) and (11), this translates into: $F(x) - 1 = 2xF(x) + 3x/(1-x)^2 \Rightarrow F(x) = \dfrac{x^2+x+1}{(1-2x)(1-x)^2} = \dfrac{A}{1-2x} + \dfrac{B}{1-x} + \dfrac{C}{(1-x)^2}$. To find A, B, and C in this partial fractions expansion, we first clear out the denominators $x^2 + x + 1 = A(1-x)^2 + B(1-x)(1-2x) + C(1-2x)$. Substituting $x = 1$ gives $C = -3$, substituting $x = 1/2$ gives $A = 7$, and substituting $x = 0$ now gives $B = -3$. Applying the expansions (8) and (11) we obtain: $F(x) = \dfrac{7}{1-2x} - \dfrac{3}{1-x} - \dfrac{3}{(1-x)^2} = 7\sum_{n=0}^{\infty}(2x)^n - 3\sum_{n=0}^{\infty}x^n - 3\sum_{n=0}^{\infty}(n+1)x^n$, and hence: $a_n = 7\cdot 2^n - 3(n+2)$.

(d) Let $F(x) = \sum_{n=0}^{\infty} a_n x^n$. $a_n = 2a_{n-2} + 5\ (n \ge 2) \Rightarrow \sum_{n=2}^{\infty} a_n x^n = 2\sum_{n=2}^{\infty} a_{n-2}x^n + 5\sum_{n=2}^{\infty} x^n =$ $2x^2\sum_{n=2}^{\infty} a_{n-2}x^{n-2} + 5\sum_{n=2}^{\infty} x^n$. Using expansion (8), this translates into: $F(x) - 1 - x = 2x^2F(x) + 5/(1-x) - 5 - 5x \Rightarrow F(x) = \dfrac{4x^2+1}{(1-2x^2)(1-x)} = \dfrac{A}{1-\sqrt{2}x} + \dfrac{B}{1+\sqrt{2}x} + \dfrac{C}{1-x}$. To find A, B, and C in this partial fractions expansion, we first clear out the denominators

$4x^2+1=A(1+\sqrt{2}x)(1-x)+B(1-\sqrt{2}x)(1-x)+C(1-2x^2)$. Substituting $x = 1$, $x=1/\sqrt{2}$, $x=-1/\sqrt{2}$, gives $C=-5, A=3/(2-\sqrt{2}), B=3/(2+\sqrt{2})$, respectively. Applying the expansion (8) we obtain: $F(x)=\frac{3/(2-\sqrt{2})}{1-\sqrt{2}x}+\frac{3/(2+\sqrt{2})}{1+\sqrt{2}x}-\frac{5}{1-x}=3/(2-\sqrt{2})\sum_{n=0}^{\infty}(\sqrt{2}x)^n$

$$+3/(2+\sqrt{2})\sum_{n=0}^{\infty}(-\sqrt{2}x)^n-5\sum_{n=0}^{\infty}x^n,$$

and hence:

$a_n=3(\sqrt{2})^n/(2-\sqrt{2})+3(-\sqrt{2})^n/(2+\sqrt{2})-5=(3/2)[2\sqrt{2}^n(1+(-1)^n)+\sqrt{2}^{n+1}(1-(-1)^n)]-5.$

(The reader is encouraged to check the correctness of this explicit formula by using the recursive one.)

(e) Let $F(x)=\sum_{n=0}^{\infty}a_nx^n$. $a_n=2a_{n-2}+a_{n-1}\ (n\ge 2)\Rightarrow \sum_{n=2}^{\infty}a_nx^n=2\sum_{n=2}^{\infty}a_{n-2}x^n+\sum_{n=2}^{\infty}a_{n-1}x^n=$ $2x^2\sum_{n=2}^{\infty}a_{n-2}x^{n-2}+x\sum_{n=2}^{\infty}a_{n-1}x^{n-1}\Rightarrow F(x)-1-2x=2x^2F(x)+x(F(x)-1)\Rightarrow F(x)=\frac{x+1}{(1-2x)(1+x)}$

$=\frac{1}{1-2x}\underset{\text{by (8)}}{=}\sum_{n=0}^{\infty}(2x)^n\Rightarrow a_n=2^n.$

#19. (a) We note that the recurrence relation $1=a_n+2a_{n-1}+3a_{n-2}+\cdots+na_1+(n+1)a_0$ remains valid when n = 0, and hence for all $n\ge 0$. Let $F(x)=\sum_{n=0}^{\infty}a_nx^n$. The recurrence relation implies that:

$$\sum_{n=0}^{\infty}x^n=\sum_{n=0}^{\infty}[a_n+2a_{n-1}+3a_{n-2}+\cdots+na_1+(n+1)a_0]x^n=\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}(k+1)a_{n-k}\right]x^n.$$

The expansion on the left is just (8), the generating function of $1/(1-x)$. By Definition 5.5 and by (11), we see that the series on the right is just the product of the generating functions $1/(1-x)^2$ and $F(x)$. Thus, the equation translates to: $\frac{1}{1-x}=\frac{F(x)}{(1-x)^2}\Rightarrow F(x)=1-x.$ The sequence defined by the recursion is thus the very simple sequence: $a_0=1, a_1=-1, a_n=0\ (n\ge 2)$. (The reader may wish to verify this directly.)

(b) We let $F(x)=\sum_{n=0}^{\infty}a_nx^n$. Unlike in part (a), the recursion formula is not valid when n = 0, it gives: $\sum_{n=1}^{\infty}nx^n=\sum_{n=1}^{\infty}\left[\sum_{k=0}^{n}(k+1)a_{n-k}\right]x^n$. The series on the left is $x\sum_{n=1}^{\infty}nx^{n-1}=$ $x\sum_{n=0}^{\infty}(n+1)x^n\underset{\text{by (11)}}{=}x/(1-x)^2$. The series on the right is the same as the corresponding series of part (a), less the zeroth term $\left[\sum_{k=0}^{0}(k+1)a_{0-k}\right]x^0=2.$ We are thus led to the equation: $x/(1-x)^2$ $=F(x)/(1-x)^2-2\Rightarrow F(x)=x+2(1-x)^2=2-3x+2x^2$. The sequence defined by the recursion is thus: $a_0=2, a_1=-3, a_2=2, a_n=0\ (n\ge 3)$. (The reader may wish to verify this directly.)

#21. (a) (i) $(1+x+x^2+x^3+x^4+x^5+x^6+x^7)^3$ Seek coefficient of x^7. (ii) 36

(b) (i) $(1+x+x^2+x^3+x^4+x^5+x^6+x^7)^2$ Seek coefficient of x^7. (ii) 8

(c) (i) $(1+x+x^2+x^3+\cdots+x^{10})^3$ Seek coefficient of x^{10}. (ii) 66

#23. (a) (i) $(1+x+x^2+x^3+x^4+x^5+x^6+x^7)\cdot(x+x^3+x^5+x^7)\cdot(x+x^2+x^3+x^4+x^5+x^6+x^7)$ Seek coefficient of x^7. (ii) 12

(b) (i) $(x+x^3+x^5+x^7)\cdot(1+x+x^2+x^3+x^4+x^5)$ Seek coefficient of x^7. (ii) 3

(c) (i) $(1+x+x^2+x^3+\cdots+x^{10})\cdot(x+x^2+x^3+\cdots+x^{10})\cdot(x^2+x^3+\cdots+x^{10})$ Seek coefficient of x^{10}. (ii) 36

#27. (a) For any positive integer k, the generating function for the number of parts of size k being used in a given partition is: $P_k(x)=1+x^k+x^{2k}+x^{3k}+\cdots$, which by (8) with the substitution $x\mapsto x^k$ can be rewritten as $P_k(x) = 1/(1-x^k)$. The generating function for $p_m(n)$ will thus be the product m of these functions: $P_1(x)\cdot P_2(x)\cdots P_m(x) = 1/(1-x)\cdot 1/(1-x^2)\cdots 1/(1-x^m)$, as asserted.

Of course, to use this generating function to compute $p_m(n)$, we would work with the polynomial form: $P_1(x)\cdot P_2(x)\cdots P_m(x) = (1+x+x^2+x^3+\cdots)(1+x^2+x^4+\cdots)\cdots(1+x^m+x^{2m}+\cdots)$, where in each parenthesized polynomial, only terms of degree at most n are listed. Thus, to compute $p(n)$, for n = 4, 5, 6, and 8, we need only look for the coefficients of x^4, x^5, x^6, x^8 in the expansion of:

$$(1+x+x^2+\cdots+x^8)(1+x^2+x^4+x^6+x^8)(1+x^3+x^6)(1+x^4+x^8)(1+x^5)(1+x^6)(1+x^7)(1+x^8).$$

Computing the indicated coefficients in this product leads us to: $p(4)=5, p(5)=7, p(6)=11$, and $p(8)=22$.

(b) The problem asks for the value of $p_5(15)$, and this will be the coefficient of x^{15} in $G_5(x)$. As explained above, we may work with the expansion:

$$(1+x+x^2\cdots+x^{15})(1+x^2+x^4+\cdots+x^{14})(1+x^3+x^6+\cdots+x^{15})(1+x^4+x^8+x^{12})(1+x^5+x^{10}+x^{15}).$$

Computing the indicated coefficient in this product leads us to: $p_5(15)=84$.

#29. (a) Since each positive integer k appears either exactly once or not at all in such a partition for n, the generating polynomial for the appearance of k is $1+x^k$, and so the generating function of $p_D(n)$ is the product of these polynomials, over all positive integers k: $(1+x)(1+x^2)(1+x^3)\cdots$.

(b) To compute $p_D(n)$ for n = 4, 5, 6, 7, and 10, we need only look for the coefficients of $x^4, x^5, x^6, x^7, x^{10}$ in the expansion of:

$$(1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)(1+x^6)(1+x^7)(1+x^8)(1+x^9)(1+x^{10}).$$

Computing the indicated coefficients in this product leads us to: $p_D(4)=2, p_D(5)=3, p_D(6)=4$, $p_D(7)=5$ and $p(10)=10$.

#31. The generating function for the sequence a_n = the number of ways to express n as a sum of distinct powers of 2 $(n \ge 1)$ is $F(x)=(1+x)(1+x^2)(1+x^4)(1+x^8)\cdots$. It suffices to show that $F(x) = 1+x+x^2+x^3+\cdots \underset{\text{by (8)}}{=} 1/(1-x)$ (i.e, this implies $a_n = 1$ for all positive indices n), and we will accomplish this by showing that $(1-x)F(x)=1$. We repeatedly apply the identity: $(x^k+1)(x^k-1) = x^{2k}-1$:

$$\begin{aligned}(1-x)F(x) &= [(1-x)(1+x)](1+x^2)(1+x^4)(1+x^8)\cdots\\ &= [(1-x^2)(1+x^2)](1+x^4)(1+x^8)\cdots\\ &= [(1-x^4)(1+x^4)](1+x^8)\cdots\\ &\vdots\\ &= 1.\end{aligned}$$

More formally, by iteratively making the substitutions $(x^{2^j}+1)(x^{2^j}-1) = x^{2\cdot 2^j}-1$, the coefficient of any fixed positive power is shown to be zero.

#35. (b) 37,917

(c) $(x+x^2+\cdots+x^9)(1+x+x^2+\cdots+x^9)^{k-1} = x(1-x^9)(1-x^{10})^{k-1}(1-x)^{-k}$